Multiple Polynomial Recurrence in Weyl Systems

Dyadisc 7

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Preliminaries

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- A nilsystem $X = G/\Gamma$ is a homogeneous space of a nilpotent Lie group G and Γ is a uniform discrete subgroup of G. The transformation T is given by left rotation by $\tau \in G$.

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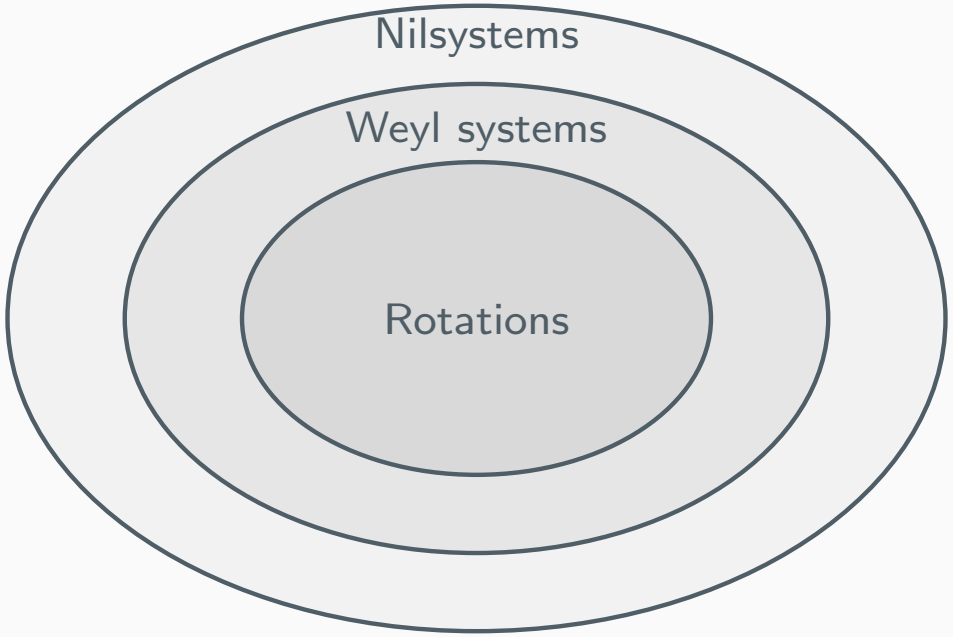
- A rotation is a nilsystem in which X is an abelian group.

Ejemplos



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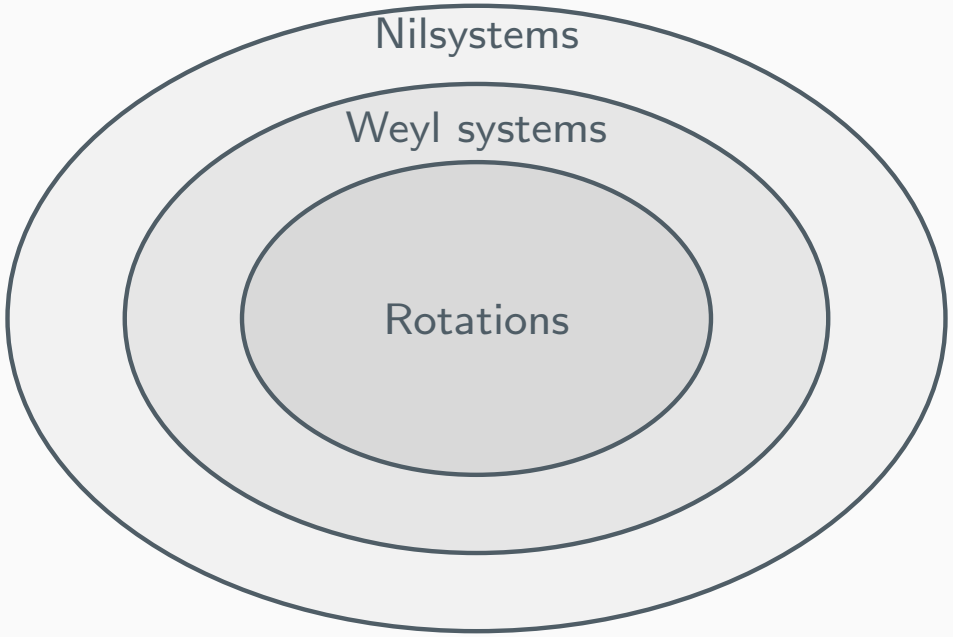
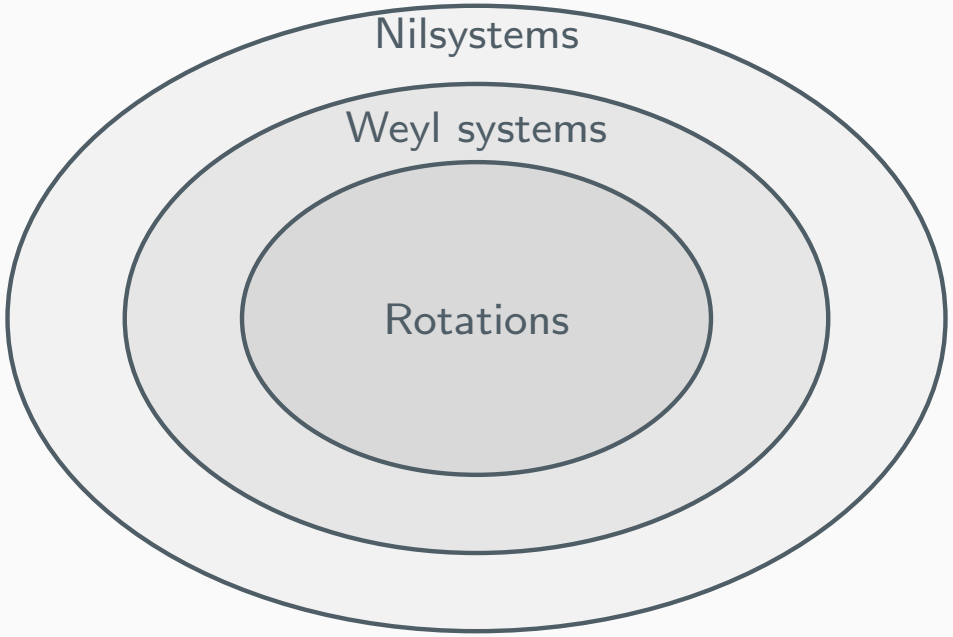


Example.

- The system $(\mathbb{T}^2, m_{\mathbb{T}^2}, T)$ where $T(x, y) = (x + \alpha, y + x)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a Weyl systems.



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- The system defined by its first coordinate, namely $(\mathbb{T}, m_{\mathbb{T}}, S)$ with $S(x) = x + \alpha$ is a rotation.

Theorem(Poincaré recurrence for sets). Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and $A \in \mathcal{B}$ with $\mu(A) > 0$. Then for infinitely many $n \in \mathbb{N}$

$$\mu(A \cap T^{-n}A) > 0.$$

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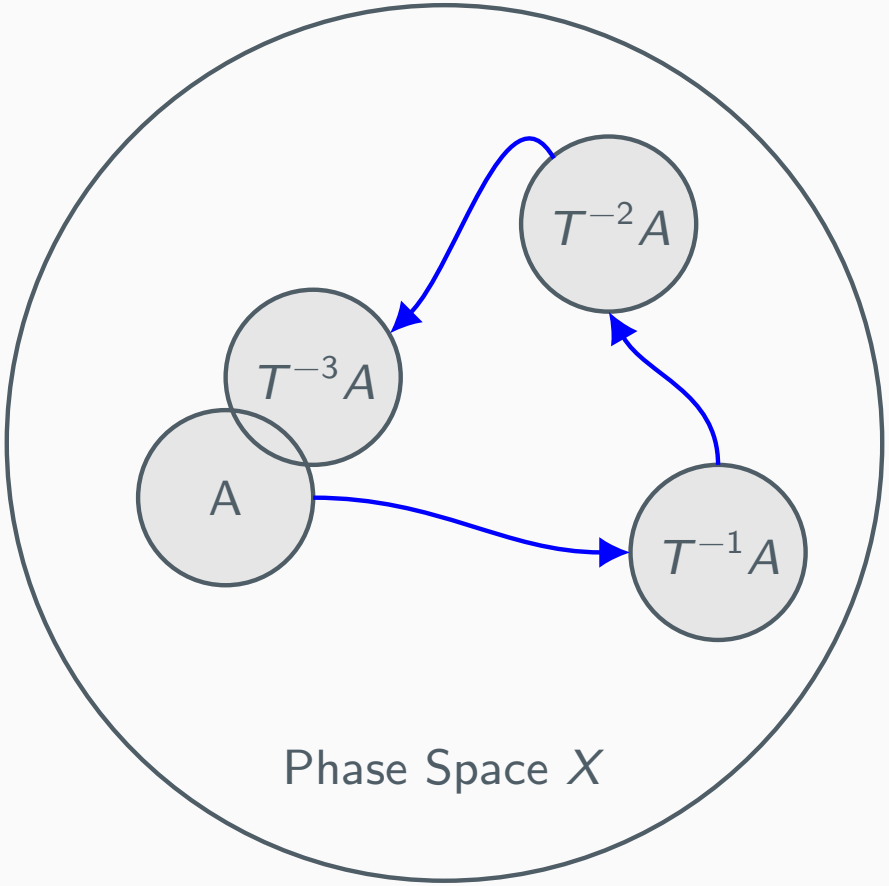

Definition. A set $R \subseteq \mathbb{Z}$ is a set of measurable recurrence if for any measure-preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, there is $n \in R$ such that

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Examples:

- $k\mathbb{N}$ for any $k \in \mathbb{N}$,
- $E \subseteq \mathbb{N}$ with $\limsup_{N\to\infty} \frac{|E\cap\{1,\dots,N\}|}{N} = 1$,
- $\mathbb{P} \pm 1$,
- $E - E$ for $E \subseteq \mathbb{N}$ infinite,

F-measurable recurrence

Definition. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. A set $R \subseteq \mathbb{Z}$ is a set of f-measurable recurrence for a family for (X, μ, T) if for $A \subseteq X$ with $\mu(A) > 0$, exists $n \in R \setminus \{0\}$ such that

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Remark: We extent this definition to a set of functions $\{f_1, \ldots, f_r\}$ by replacing $\mu(A \cap T^{-f(n)} A) > 0$ by

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Example.If $f_1(n) = 2n$, $f_2(n) = n^2$, then R is a set of $\{2n, n^2\}$-measurable recurrence for a system (X, μ, T) if for each $A \in \mathcal{B}(X)$ with $\mu(A) > 0$ there is $n \in R$ with

$$\mu(A \cap T^{-2n} A \cap T^{-n^2} A) > 0.$$

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Question. Let

- P and Q be sets of polynomials,
- $\mathcal{F}_1$ and $\mathcal{F}_2$ sets of measure preserving systems or minimal systems, and
- $\mathcal{R}_1$ the family of sets of P-recurrence for $\mathcal{F}_1$ and $\mathcal{R}_2$ the family of sets of Q-recurrence for $\mathcal{F}_2$.

When is it possible to establish either

$$\mathcal{R}_1 \subseteq \mathcal{R}_2, \text{ or } \mathcal{R}_1 \setminus \mathcal{R}_2 \neq \emptyset \text{ and } \mathcal{R}_2 \setminus \mathcal{R}_1 \neq \emptyset?$$

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If

$$\mathcal{R}_1 \subseteq \mathcal{R}_2$$

We say that $\mathcal{R}_1$ implies ($\Rightarrow$) $\mathcal{R}_2$.

If

$$\mathcal{R}_1 \setminus \mathcal{R}_2 \neq \emptyset \text{ and } \mathcal{R}_2 \setminus \mathcal{R}_1 \neq \emptyset$$

we say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are in general positions ($\perp\!\!\!\perp$).

Some important results

1. Folklore: For integral polynomials $P = \{p_1, \ldots, p_r\}$,

P-measurable recurrence for rotations $\iff$ P-topological recurrence for rotations

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6. [4, 2024] Griesmer:

   $\{n, 2n\}$-measurable recurrence $\perp\!\!\!\perp$ $\{n^2\}$-measurable recurrence

Results in Weyl systems

Theorem(H. 2024+). Let $\{p_1, ..., p_r\}$ polynomials. Then, there exist rationally independent polynomials $\{q_1, ..., q_r\}$ such that the following are equivalent:

1. R is a set of $\{q_1, \dots, q_r\}$-recurrence for rotations.
2. R is a set of $\{p_1, \dots, p_r\}$-measurable recurrence for Weyl systems,
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Remark: A set $R \subseteq \mathbb{Z}$ is of $\{q_1, \dots, q_r\}$-recurrence for rotations if for each $\alpha \in \mathbb{T}^d$ and $\epsilon > 0$ there is $n \in R$ such that

$$\|q_1(n)\alpha\|_{\mathbb{T}^d} < \epsilon, \dots, \|q_r(n)\alpha\|_{\mathbb{T}^d} < \epsilon.$$

Model Example

Example. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

1. For $P_1 = \{n, 2n, 3n\}$ we have $Q_1 = \{n, n^2, n^3\}$, and
2. for $P_2 = \{n, 2n, n^2\}$ we have $Q_2 = \{n, n^2, 2n^3 - n^4\}$.

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For distinguishing these two notions, we look at

$$\left\{ n \in \mathbb{N} : \|n^3 \alpha\|_{\mathbb{T}} > 1/4 \right\}$$

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$$\left\{ n \in \mathbb{N} : \|(2n^3 - n^4)\alpha\|_{\mathbb{T}} > 1/4 \right\},$$

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Definition. Let (X, μ, T) be a system and (Y, ν, T) a factor. We say that (Y, ν, T) is characteristic for the ergodic averages over the patterns $P = \{p_1, \dots, p_r\}$ if:

For any set of functions $f_1, \dots, f_r \in L^\infty(X)$

$$\lim_{N-M\to\infty} \| \frac{1}{N-M} \sum_{n=M+1}^{N} f_1\left(T^{p_1(n)}x\right) \cdots f_r\left(T^{p_r(n)}x\right) - \frac{1}{N-M} \sum_{n=M+1}^{N} \mathbb{E}(f_1 \mid Y)\left(T^{p_1(n)}x\right) \cdots \mathbb{E}(f_r \mid Y)\left(T^{p_r(n)}x\right) \|_{L^2(X)} = 0.$$

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Definition. Let $P = \{p_1, \dots, p_r\}$ polynomials. We define the Weyl complexity $W(P)$ as the minimal k such that for every Weyl system (X, μ, T), its maximal k-step nilfactor (Y, ν, T) is characteristic for the ergodic averages over the patterns P.

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For a polynomial p and $k \in \mathbb{N}$ denote $p^{[k]}(n) = \binom{p(n)}{k}$. Define for integral polynomials $P = \{p_1, \ldots, p_r\}$ and $k = W(P)$,

$$\Lambda(P) = \begin{bmatrix} p_1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_r & 0 & 0 & \ldots & 0 & 0 \\ p_1^{[2]} & p_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_r^{[2]} & p_r & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_1^{[k]} & p_1^{[k-1]} & p_1^{[k-2]} & \ldots & p_1^{[2]} & p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_r^{[k]} & p_r^{[k-1]} & p_r^{[k-2]} & \ldots & p_r^{[2]} & p_r \end{bmatrix}$$

A standard Weyl system (X, T) is $X = \mathbb{T}^k$ such that for each $x = (x_1, \ldots, x_k) \in X$

$$T(x_1, x_2, \ldots, x_k) = (x_1 + \alpha, x_2 + x_1, x_3 + x_2, \ldots, x_k + x_{k-1}).$$

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In particular

$$T^n(x_1, x_2, \ldots, x_k) = \left(x_1 + n\alpha, x_2 + nx_1 + \binom{n}{2}\alpha, \cdots, x_d + \sum_{i=1}^{k-1} \binom{n}{i} x_{k-i} + \binom{n}{k}\alpha \right).$$

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After reordering coordinates, $(T^{p_1(n)}x - x, \ldots, T^{p_r(n)}x - x)$ corresponds to

$$\Lambda(P) \begin{bmatrix} \alpha \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

Fix $k = W(P)$.

Definition. (Weyl Space) We define

$$\mathcal{WS}(P) = \left\{ q(n) \mid \exists v \in \mathbb{R}^{r \cdot k},\ v^T \Lambda_k(P) = (q(n), 0, \ldots, 0) \right\}.$$

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Example.

- $W(n, 2n, 3n) = 3$ and $\mathcal{WS}(n, 2n, 3n) = \langle n, n^2, n^3 \rangle_{\mathbb{R}}$,

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- $W(n, 2n, 3n) = 3$ and $\mathcal{WS}(n, 2n, 3n) = \langle n, n^2, n^3 \rangle_{\mathbb{R}}$,
- $W(n, 2n, n^3) = 2$ and $\mathcal{WS}(n, 2n, n^3) = \langle n, n^2, n^3 \rangle_{\mathbb{R}}$,

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2. If $W(P) = 1$ then $\mathcal{WS}(P) = \langle P \rangle_{\mathbb{R}}$.
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3. If all polynomials in P are linear, then $\mathcal{WS}(P)$ is the set of integral polynomial of degree at most $|P|$.
   Example: If $P = \{n, 11n, 42n\}$ then $\mathcal{WS}(P) = \langle n, n^2, n^3 \rangle_{\mathbb{R}}$.

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4. $\mathcal{WS}(\{p_1 \circ q, \ldots, p_r \circ q\}) = \mathcal{WS}(\{p_1, \ldots, p_r\}) \circ q$.
   Example: If $P = \{p(n), 2p(n), 3p(n)\}$ then $\mathcal{WS}(P) = \langle p(n), p(n)^2, p(n)^3 \rangle_{\mathbb{R}}$.

Lemma

- *Let $P = \{p_1, \dots, p_r\}$ be a family of integral polynomials.*
- *Let (X, μ, T) be a product of s many standard d-step Weyl systems, this is $X = \mathbb{T}^{sd}$ and T decomposes as $T = S_1 \times \cdots \times S_s$ where $(\mathbb{T}^d, S_i)$ is a standard Weyl system such that for $b_i \in \mathbb{R} \setminus \mathbb{Q}$,*

$$S_i(x_1, ..., x_d) = (x_1 + b_i, x_2 + x_1, ..., x_d + x_{d-1}).$$

Then for all functions $f_0, \dots, f_r \in L^\infty(\mu)$ there are a sequence $(c_l)_{l \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ and integral polynomials $(q_l)_{l \in \mathbb{Z}} \in \mathcal{WS}(P)^s$ such that

$$\int_X f_0 \, T^{p_1(n)} f_1 \cdots T^{p_r(n)} f_r \, d\mu = \sum_{l \in \mathbb{Z}} c_l \cdot e\left(q_l(n)^T b\right).$$

Lifting from Weyl systems to all systems

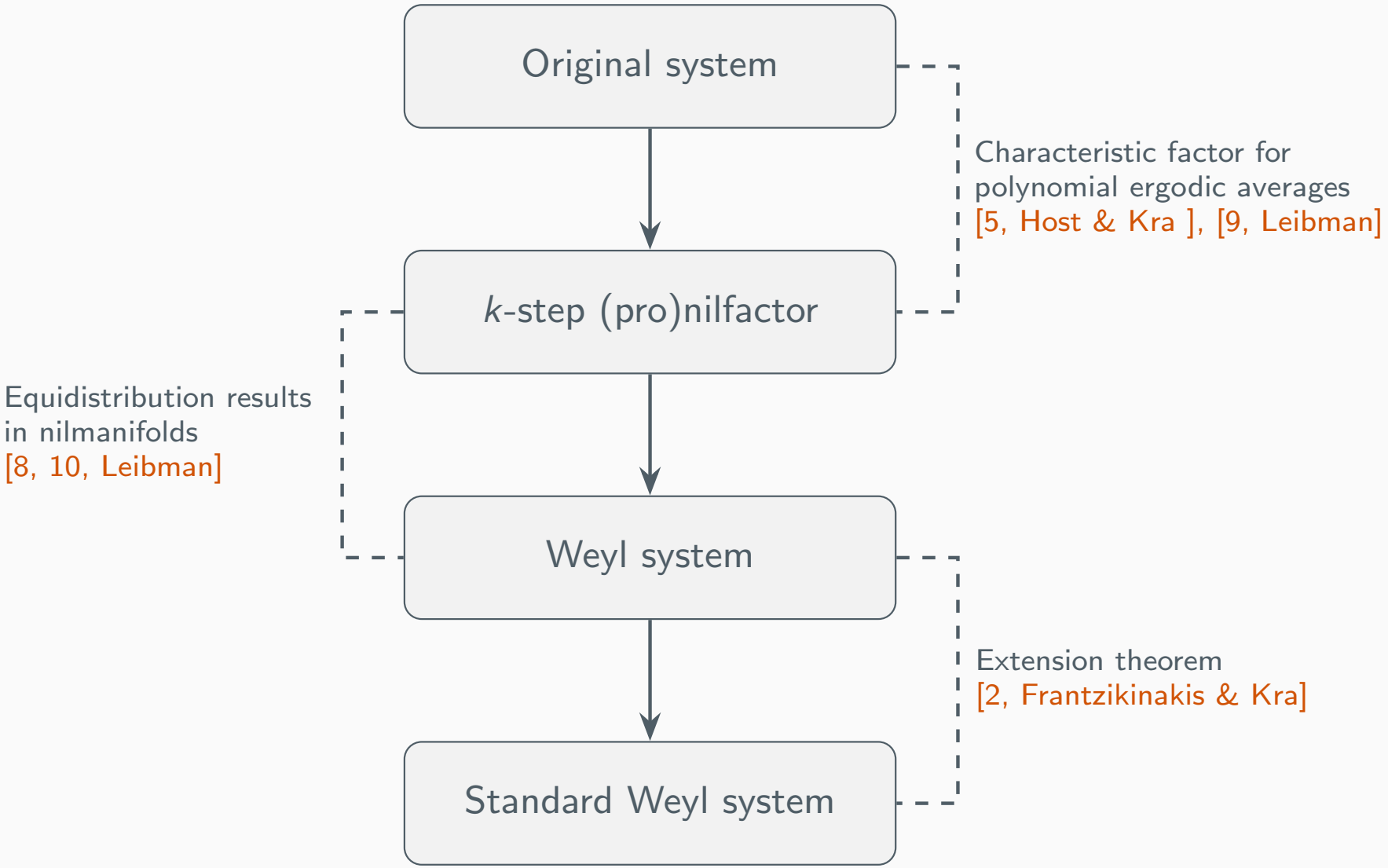
Theorem(H. 2024+).

- Let (X, μ, T) be an invertible measure preserving system,
- let $P = \{p_1, \dots, p_r\}$ be a set of integral polynomials, h a integral polynomial such that $h \notin \mathcal{WS}(P)$, and
- let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

If $g : \mathbb{T} \to \mathbb{C}$ is Riemann integrable and $f_0, f_1, \dots, f_r \in L^\infty(\mu)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(g(h(n)\alpha) - \int_{\mathbb{T}} g d\lambda \right) \int f_0 T^{p_1(n)} f_1 \cdots T^{p_r(n)} f_r d\mu = 0$$

Strategy


Original system
k-step (pro)nilfactor
Weyl system
Standard Weyl system
Characteristic factor for polynomial ergodic averages [5, Host & Kra], [9, Leibman]
Equidistribution results in nilmanifolds [8, 10, Leibman]
Extension theorem [2, Frantzikinakis & Kra]

Sketch of the proof

By approximating by exponentials, we can assume that $g(t) = e(kt)$ for $t \in \mathbb{Z}$.

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$$\left| \int f_0 T^{p_1(n)} f_1 \cdots T^{p_r(n)} f_r \, d\mu - \sum_{l \in [-M,M]} c_l \cdot e\left(q_l(n)^T b\right) \right| \leq \epsilon.$$

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Thus, we have

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{n=1}^{N} e(kh(n)\alpha) \cdot \int f_0 T^{p_1(n)} f_1 \cdots T^{p_r(n)} f_r d\mu \right| \\
&\leq \epsilon + \left| \frac{1}{N} \sum_{n=1}^{N} \sum_{l \in [-M,M]} c_l \cdot e\left(kh(n)\alpha + \sum_{j=1}^{s} q_l^j(n) \cdot b_j \right) \right| \\
&\leq \epsilon + \sum_{l \in [-M,M]} \left| \frac{1}{N} \sum_{n=1}^{N} e\left(kh(n)\alpha + \sum_{j=1}^{s} q_l^j(n) \cdot b_j \right) \right|.
\end{aligned}$$

By Weyl's equidistribution criterion it is enough to see that the following polynomial has an irrational coefficient:

$$\underbrace{kh(n)\alpha}_{\text{External frequency}} + \underbrace{\sum_{j=1}^{s} q_l^j(n) \cdot b_j}_{\mathcal{WS}(P)\text{-frequencies}} .$$

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This follows from the fact that $h \notin \mathcal{WS}(P)$ and $\{1, b_1, \ldots, b_s\}$ are l.i.. That being so,

$$\limsup_N \left| \frac{1}{N} \sum_{n=1}^{N} e(kh(n)\alpha) \cdot \int f_0 T^{p_1(n)} f_1 \cdots T^{p_r(n)} f_r \, d\mu \right| \leq \epsilon,$$

Taking $\epsilon \searrow 0$ finishes the proof. □

Results in all systems

Theorem(H. 2024+). Let $P = \{p_1, ..., p_k\}$ and $Q = \{q_1, ..., q_l\}$ two sets of polynomials such that $\mathcal{WS}(P) \setminus \mathcal{WS}(Q) \neq \emptyset$. Then

$$Q\text{-measurable recurrence} \not\Rightarrow P\text{-measurable recurrence}$$

In particular, if in addition $\mathcal{WS}(Q) \setminus \mathcal{WS}(P) \neq \emptyset$ then

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The set S is not a set of P-recurrence, but it is a set of (averaging) Q-recurrence: Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system, and $A \subseteq X$ with $\mu(A) > 0$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mathbb{1}_S(n) \mu(A \cap T^{-q_1(n)} A \cap \cdots \cap T^{-q_r(n)} A)$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(A \cap T^{-q_1(n)} A \cap \cdots \cap T^{-q_r(n)} A) > 0.$$

Thanks for your attention!
Any questions?

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