Asymptoticity, Automorphism Group and Strong Orbit Equivalence

Haritha Cheriyath
Based on an ongoing work with Sebastián Donoso

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$$\implies \phi(\{T^n(x) \mid n \in \mathbb{Z}\}) = \{S^n(\phi(x)) \mid n \in \mathbb{Z}\}$$

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- There exists $\alpha, \beta : X \to \mathbb{Z}$ such that

$$\phi(T^{\alpha(x)}(x)) = S(\phi(x)) \text{ and } \phi(T(x)) = S^{\beta(x)}(\phi(x))$$

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- (X, T) and (Y, S) are strong orbit equivalent if α, β are discontinuous at most at one point

Conjugacy invariants under strong orbit equivalence

We restrict our study to minimal Cantor systems!

- Herman Putnam and Skau 1992: Invariant measure is preserved under SOE
- Sugisaki 2007: For any $\alpha \geq 0$ and a SOE class, there is a subshift with topological entropy α (Boyle & Handelman 1994)
- Ormes 1997: If two systems are strong orbit equivalent, then they share the same continuous rational eigenvalues. But not true in general (Cortez, Durand, Maass, Frank ...)

Automorphism groups

- An *automorphism* on (X, T) is a homeomorphism $\phi : X \to X$ such that $\phi \circ T = T \circ \phi$
- $\mathrm{Aut}(X, T)$ is a group under composition
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Results

- Boyle, Lind, and Rudolph (1988): subshift of finite type contains various subgroups
- Hochman (2010): any SFT (multidimensional) with positive entropy admits any finite group in its automorphism group
- Cyr and Kra (2016): For X minimal and $\limsup_{n\to\infty} \frac{p_X(n)}{n^d}$ for some $d \in \mathbb{N}$, $\mathrm{Aut}(X, S)$ is amenable
- Donoso, Durand, Maass and Petite (2016) For X minimal and $\liminf_{n\to\infty} \frac{p_X(n)}{n} < \infty$, $\mathrm{Aut}(X, S)$ is virtually $\mathbb{Z}$

Asymptotic components

- $x, y \in X$ is said to be *left asymptotic* with respect to T if $\lim_{n\to+\infty} \text{dist}(T^{-n}x, T^{-n}y) = 0$
- For subshifts, if x, y are left asymptotic, then there exists $\ell \in \mathbb{Z}$ such that $x_{(-\infty,\ell)} = y_{(-\infty,\ell)}$
- $\text{Orb}_T(x)$ is asymptotic to $\text{Orb}_T(y)$, if there exist $x' \in \text{Orb}_T(x)$ and $y' \in \text{Orb}_T(y)$ that are asymptotic
- When an equivalent class is not reduced to a single element, we call it an *asymptotic component*
- Elements on the automorphism group permutes the asymptotic components

Proposition

If (X, T) is a minimal topological dynamical system with exactly one asymptotic component, then $Aut(X, T)$ is generated by T.

Results

Theorem

Let (X, T) be a topological dynamical system with at most countably many asymptotic components. Then (X, T) has topological entropy zero.

Idea of Proof:

- Blanchard, Host and Ruette 2005: when $h_\mu(T) > 0$, then

$$\mu\left(\{x \in X \mid x \sim y \text{ for } y \neq x\}\right) = 1$$

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Corollary

Let (X, T) be a minimal Cantor system. Then there exists a minimal subshift that is strong orbit equivalent to (X, T) and having uncountably many asymptotic components.

Results

Theorem (C., Donoso)

For any given $k \geq 1$ and for any minimal Cantor system (X, T), there exists a minimal subshift which has exactly k many asymptotic components and that is SOE to (X, T).
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Corollary

Let (X, T) be a minimal Cantor system. Then there exists a minimal subshift (X', S) that is strong orbit equivalent to (X, T) and $Aut(X', S)$ is isomorphic to $\mathbb{Z}$.

Tools - (i) Symbolic dynamics

- $(\mathcal{A}_n)_{n\geq 0}$ be a sequence of finite alphabets and $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be a directive sequence of morphisms

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$$X_{\boldsymbol{\tau}}^{(n)} := \{x \in \mathcal{A}_n^{\mathbb{Z}} \mid \forall\ k \geq 0,\ x_{[-k,k]} \prec \tau_{[n,N)}(a) \text{ for some } N > n \text{ and } a \in \mathcal{A}_N\}$$

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- $X_{\boldsymbol{\tau}} := X_{\boldsymbol{\tau}}^{(0)}$, the *$\mathcal{S}$-adic subshift generated by $\boldsymbol{\tau}$*
- If $\boldsymbol{\tau}$ is primitive, then $X_{\boldsymbol{\tau}}^{(n)}$ is non-empty and minimal for all $n \geq 0$

Recognizability

$\boldsymbol{\tau}$ is recognizable if for each $n \geq 0$ and $x \in X_{\boldsymbol{\tau}}$, there exists a unique tuple (k, y) where $y \in X_{\boldsymbol{\tau}}^{(n)}$ and $0 \leq k < |\tau_{[0,n)}(y_0)|$ such that $x = S^k(\tau_{[0,n)}(y))$

In this case we say that y is the $\tau_{[0,n)}$- *factorization* of x.

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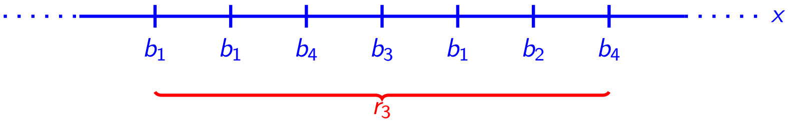

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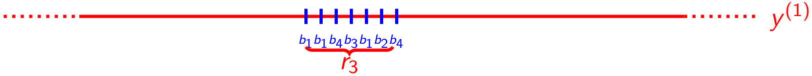

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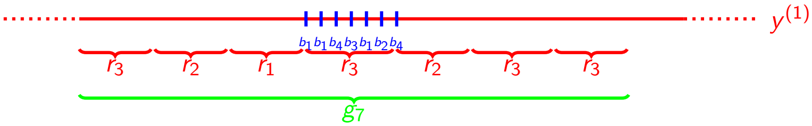

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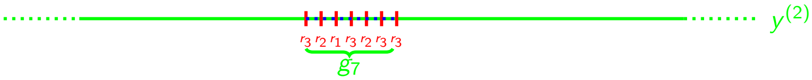

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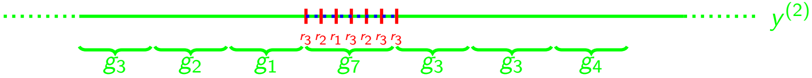

Local property of recognizability

If $X_{\boldsymbol{\tau}}$ is recognizable, then for all $n \geq 0$, there exists a $R > 0$ such that whenever $x, x' \in X_{\boldsymbol{\tau}}$ are such that $x_{[-R,R]} = x'_{[-R,R]}$, then $y_0 = y'_0$ where $y, y' \in X_{\boldsymbol{\tau}}^{(n)}$ are the $\tau_{[0,n)}$-factorizations of x and x' respectively.

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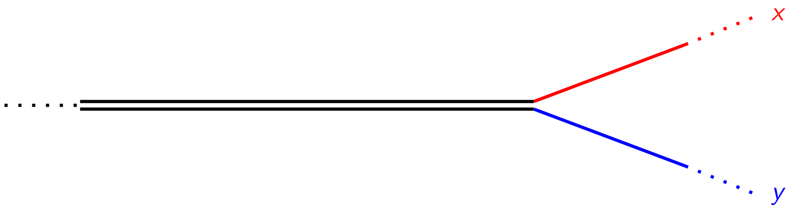

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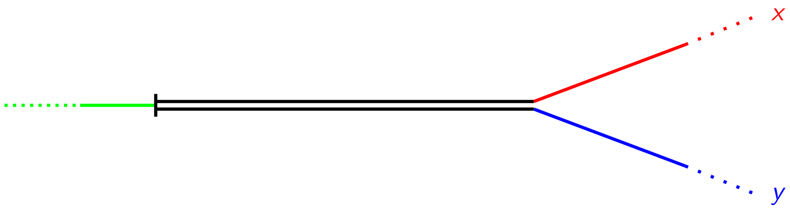

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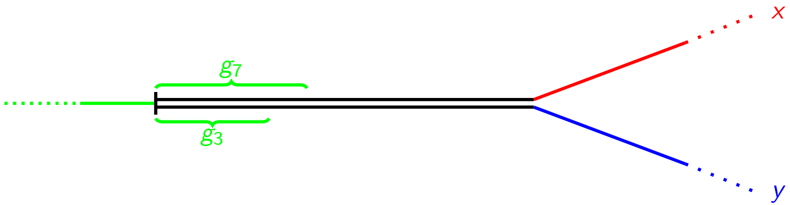

$x^{(n)}, y^{(n)} \in X_{\boldsymbol{\tau}}^{(n)}$ are asymptotic and $\tau_{[0,n)}(g_3)$ is a prefix of $\tau_{[0,n)}(g_7)$

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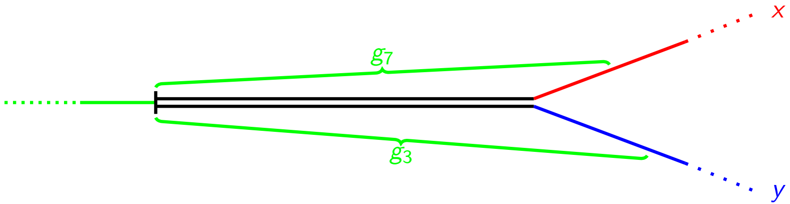

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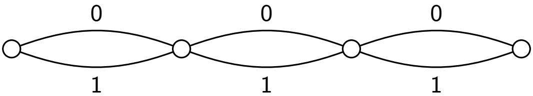
Tools - (ii) Bratteli Vershik representation

Example (Odometer): $(\{0,1\}^{\mathbb{N}}, T)$ where $T(x) = x + (100\dots)$ with carrying over

$$T(11\dots10\dots) = 00\dots01\dots \text{ and } T(1_\infty) = 0_\infty$$



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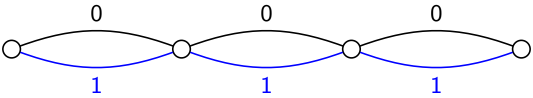

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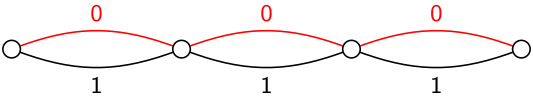

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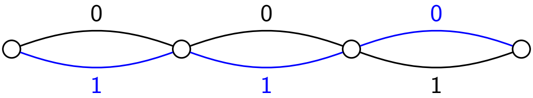

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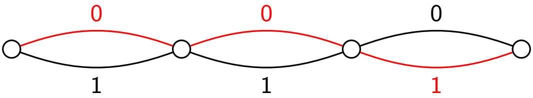

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A Bratteli diagram is a directed infinite graph $B = (V, E)$ where

- Both V and E are partitioned into non-empty finite levels as $V = \bigcup_{n\geq 0} V_n$ and $E = \bigcup_{n\geq 1} E_n$.
- $V_0 = \{v_0\}$ and all edges in E_n, for $n \geq 1$, are from V_{n-1} to V_n.

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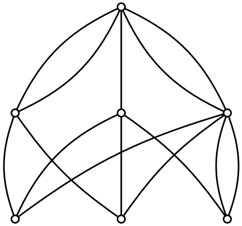

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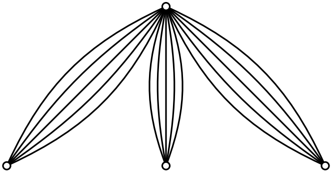
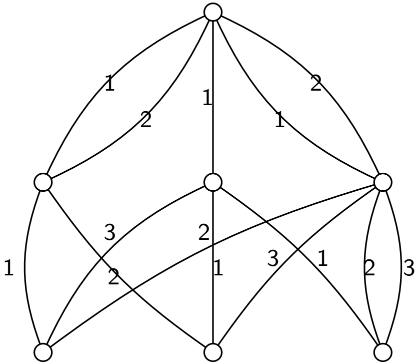


(V, E) is *simple*, if there exists a telescoping of (V, E) such that any two vertices on different levels are connected by a path.

Ordered Bratteli - diagram and Vershik map

- We give a partial order $\leq$ on E where $e, e' \in E$ are comparable if and only if they have the same range
- Induces a partial order on finite paths on (V, E)



Ordered Bratteli-diagram and Vershik map

- Space: $X_B = \{e_1 e_2 \dots \ | \ e_i \in E_i \text{ and } r(e_i) = s(e_{i+1}) \text{ for } i \geq 1\}$
- $B = (V, E, \leq)$ is *properly ordered* if it is simple and contain exactly one minimal and maximal point
- $T_B(x_{\max}) = x_{\min}$

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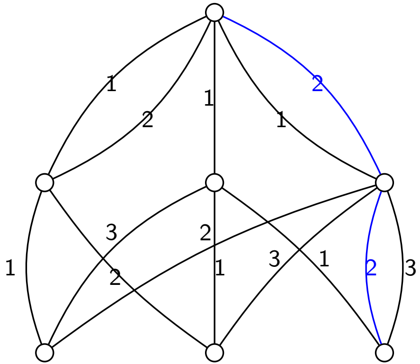

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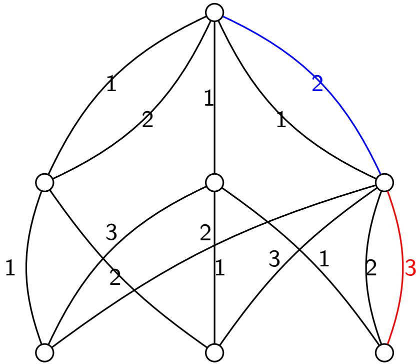

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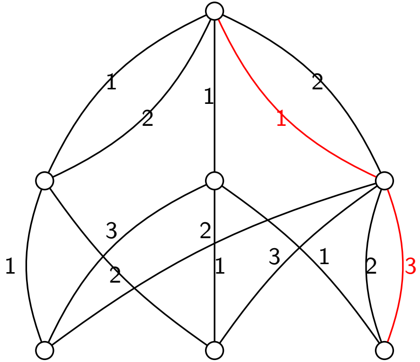

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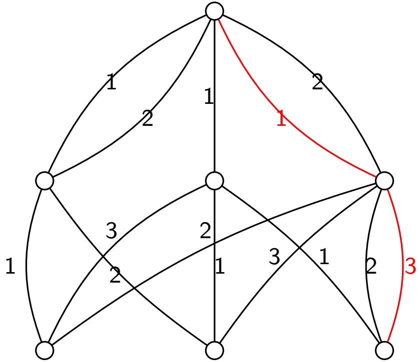

- *Bratteli-Vershik system*: (X_B, T_B) is a minimal Cantor system When $B = (V, E, \leq)$ is properly ordered

Bratteli Vershik representation

Theorem (Herman, Putnam and Skau 1992)

Let (X, T) be a minimal Cantor system. Then there exists a properly ordered Bratteli diagram $B = (V, E, \leq)$ such that (X_B, T_B) is topologically conjugate to (X, T).

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Theorem (Herman, Putnam and Skau 1992)

Let (X, T) be a minimal Cantor system. Then there exists a properly ordered Bratteli diagram $B = (V, E, \leq)$ such that (X_B, T_B) is topologically conjugate to (X, T).

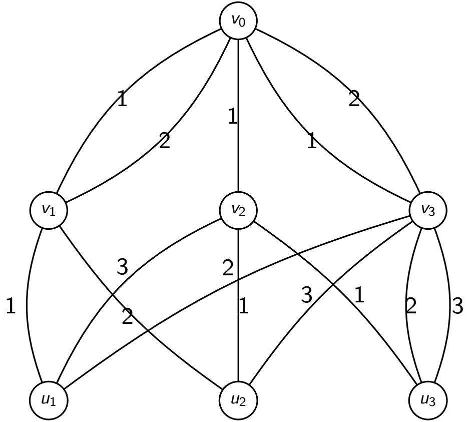
Theorem (Giordano, Putnam, and Skau 1995)

Let (X_1, T_1) and (X_2, T_2) be two minimal Cantor systems. Then (X_1, T_1) and (X_2, T_2) are strong orbit equivalent if and only if the associated ordered Bratteli diagrams have a common intertwining.

In particular, they are strong orbit equivalent if they have the same Bratteli diagram without ordering.

Morphisms read on an ordered Bratteli-diagram

- For a given $B = (V, E, \leq)$, we define a sequence $\boldsymbol{\tau}_B = (\tau_n)_{n\geq 0}$ of morphisms
- $u \in V_{n+1}$, let $(e_1, \dots, e_k)$ be an ordered list such that $\{e_1, \dots, e_k\} = r^{-1}(u)$
- For $n \geq 1$, $\tau_n : V_{n+1}^* \to V_n^*$, as $\tau_n(u) = s(e_1) \dots s(e_k)$
- $\tau_0 : V_1^* \to E_1^*$ as $\tau_0(u) = e_1 \dots e_k$



$\tau_1(u_1) = v_1 v_3 v_2$ and $\tau_1(u_3) = v_2 v_3^2$

Morphisms read on an ordered Bratteli-diagram

- For a given $B = (V, E, \leq)$, we define a sequence $\boldsymbol{\tau}_B = (\tau_n)_{n \geq 0}$ of morphisms
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Proposition

Let $B = (V, E, \leq)$ be an ordered Bratteli diagram and $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ be the directive sequence of morphisms read on B. Suppose that $X_{\boldsymbol{\tau}}$ is minimal, τ_n is proper for $n \geq 1$ and each τ_n extends by concatenation to an injective map from $X_{\boldsymbol{\tau}}^{(n+1)}$ to $X_{\boldsymbol{\tau}}^{(n)}$ for $n \geq 0$. Then $(X_{\boldsymbol{\tau}}, S)$ is conjugate to (X_B, T_B).

Recap

$$
\begin{tikzcd}
(X, T) \arrow[r, leftrightarrow, "\text{Conjugacy}"] \arrow[d, leftrightarrow, "\text{SOE}"'] & (X_B, T_B) \arrow[d, leftrightarrow, "\text{SOE}"] \\
(X_{T_{B'}}, S) \arrow[r, leftrightarrow, "\text{Conjugacy}"'] & (X_{B'}, T_{B'})
\end{tikzcd}
$$

Construction of subshifts with one asymptotic component

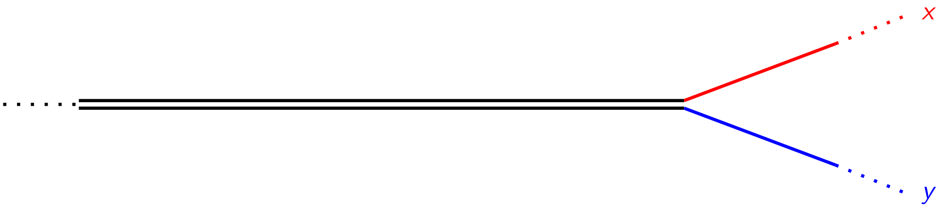
Definition

Let $\boldsymbol{\tau} = (\tau_n : V_{n+1}^* \to V_n^*)_{n \geq 0}$ satisfies the following conditions: τ_0 is a hat morphism and for $n \geq 1$,

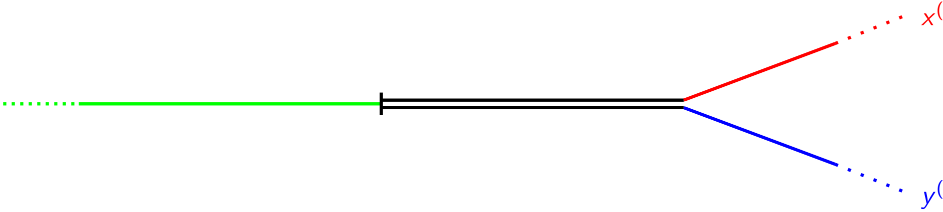
1. τ_n is primitive,
2. $\tau_n(u_i) = v_1^i v_2 \ldots\ldots v_m$, and it does not have $v_m v_1$ as a subword.

- τ_n is injective on symbols: there does not exist $u \neq u' \in V_{n+1}$ such that $\tau_n(u) = \tau_n(u')$
- $X_{\boldsymbol{\tau}}$ is recognizable
- There does not exist $u \neq u' \in V_{n+1}$ such that $\tau_n(u)$ and $\tau_n(u')$ have the same prefix $v_1^i v_2$

Asymptotic pairs



Asymptotic pairs



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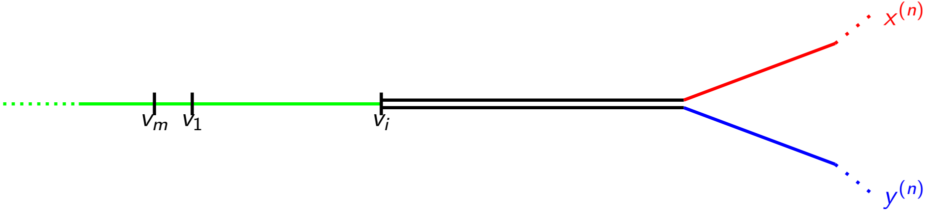

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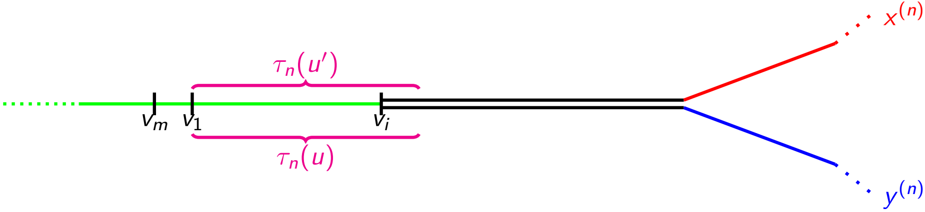

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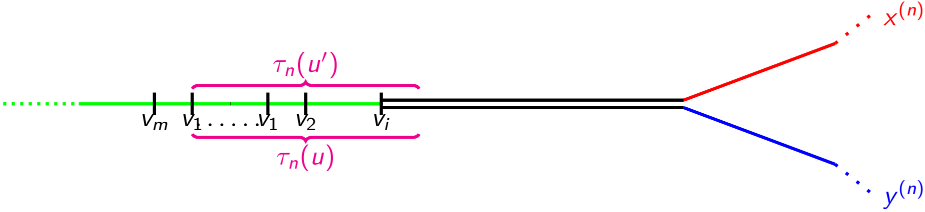

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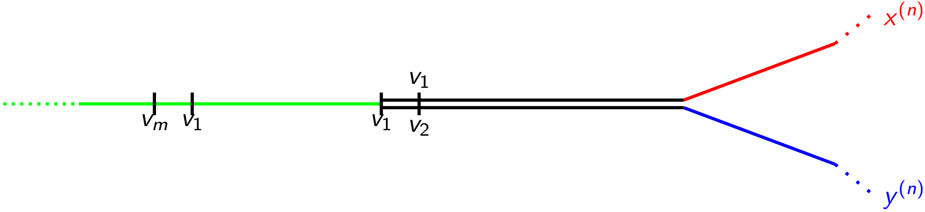

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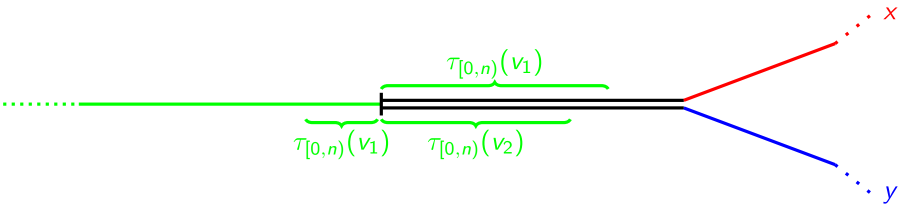

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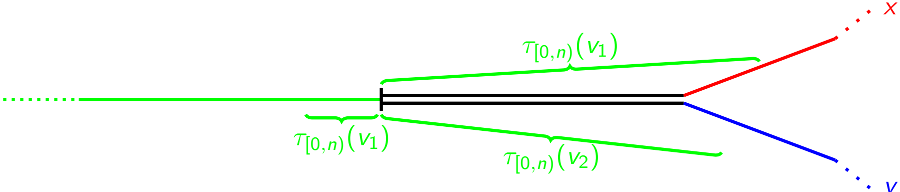
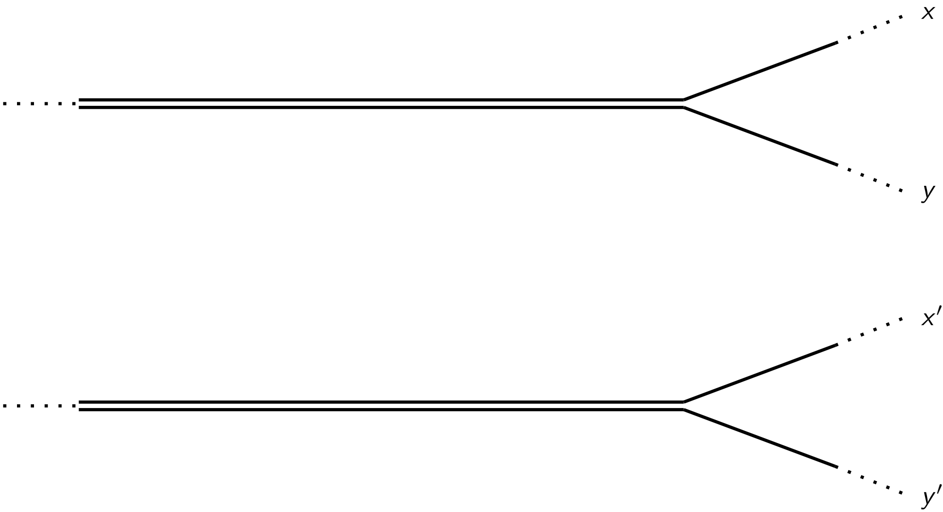


Only one asympotic component



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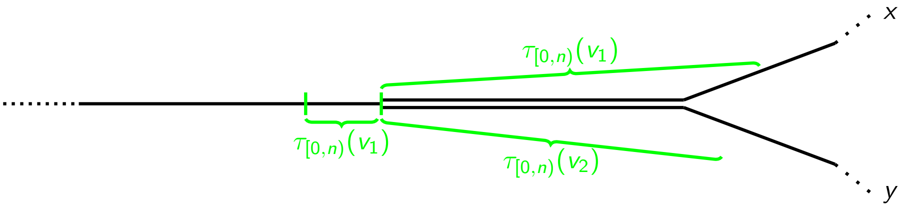

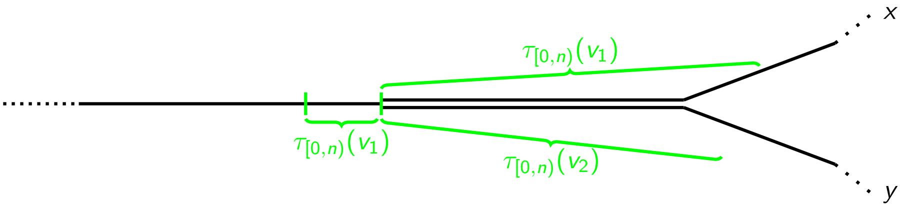

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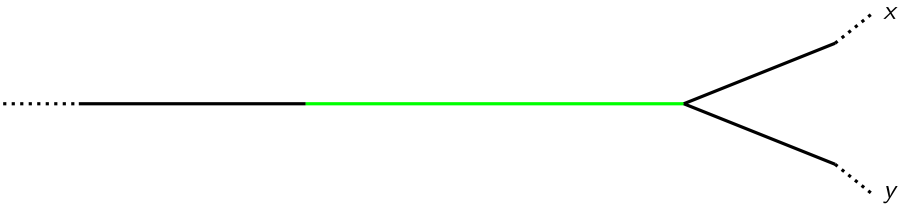

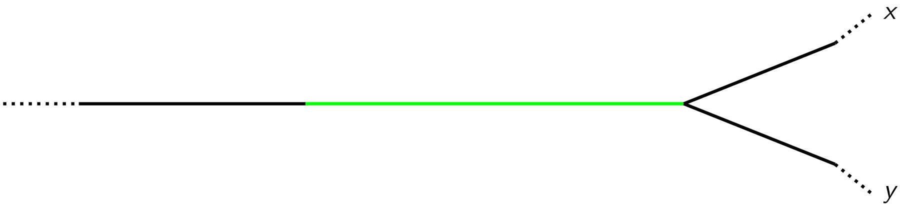

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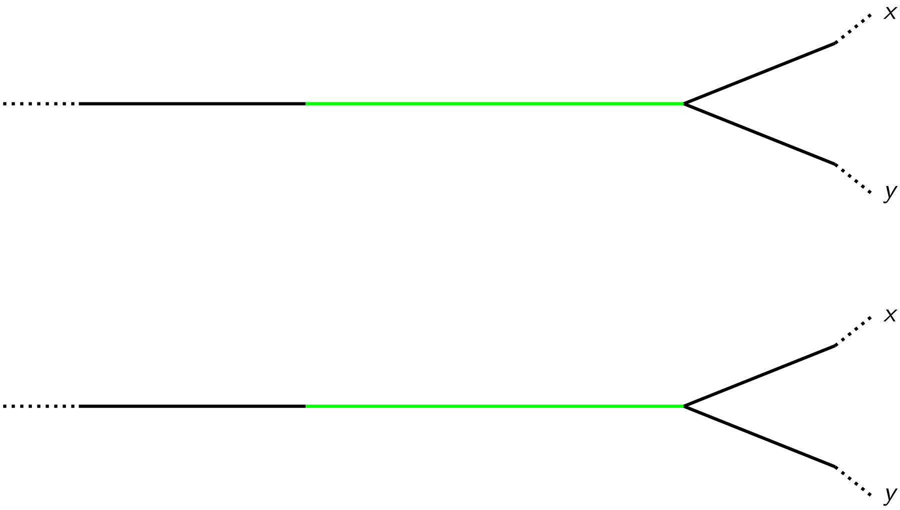

$|\tau_{[0,n)}(v_1)| \to \infty$ as $n \to \infty$

Finitely many asymptotic components

Fix $k \geq 1$. Then $\tau_n : V_{n+1}^* \to V_n^*$ is defined as

$$
\begin{array}{ccl}
u_1 & \to & v_1 v_2 v_3 \ldots v_k v_{k+1} \ldots\ldots\ldots v_m \\
u_2 & \to & v_1^2 v_2 v_3 \ldots v_k v_{k+1} \ldots\ldots\ldots v_m \\
u_3 & \to & v_1^2 v_2^2 v_3 \ldots v_k v_{k+1} \ldots\ldots\ldots v_m \\
\vdots & & \vdots \\
u_{k+1} & \to & v_1^2 v_2^2 v_3^2 \ldots v_k^2 v_{k+1} \ldots\ldots\ldots v_m \\
u_{k+2} & \to & v_1^{k+2} v_2 \ldots\ldots\ldots v_m \\
\vdots & & \vdots \\
u_t & \to & v_1^t v_2 \ldots\ldots\ldots v_m
\end{array}
$$

More results

Corollary

Let K be any compact metrizable Choquet simplex. Then there exists a subshift (Y, S) with trivial automorphism group such that its set of invariant measures, $\mathcal{M}(Y, S)$ is affine homeomorphic to K.

Downarowicz 1991: There exists a subshift (X, S) such that $\mathcal{M}(X, S)$ is affine homeomorphic to K

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Let (X, T) be a minimal Cantor system and $(p_n)_{n \geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{n}{p_n} = 0$. Then there exists a subshift (Y, S) that is strong orbit equivalent to (X, T), $\lim_{n\to\infty} \frac{p_Y(n)}{p_n} = 0$, and $Aut(Y, S) \cong \mathbb{Z}$

Cecchi Bernales and Donoso 2024: Constructed a subshift that satisfy the first two properties

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Sub-exponential complexity with one asymptotic component?