Almost 1-1 extensions of Furstenberg-Weiss type: Test for amenability.

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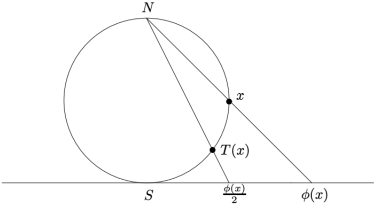

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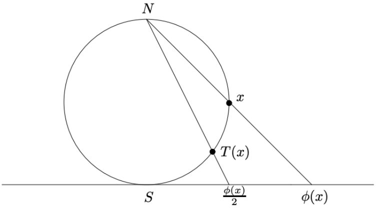

The set of invariant probability measures of T is
$\{t\delta_N + (1-t)\delta_S : t \in [0,1]\}$.

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Let $\langle T, T_\alpha \rangle$ be the subgroup of $(\mathrm{Homeo}(S^1), \circ)$ generated by T_α and T.

- $\langle T, T_\alpha \rangle$ acts by homeomorphisms on S^1.
- Remember that μ is an invariant probability measure for this action if for every $g \in \langle T, T_\alpha \rangle$ and every Borel set $A \subseteq S^1$,

$$\mu(gA) = \mu(A)$$

- In particular, if μ is invariant for this action, then μ is invariant for T_α and T.
- $\Longrightarrow$ the space of invariant probability measures of this action is empty!

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General setting and question.

We deal with **dynamical systems** (X, G) such that :

- X is a compact metric space.
- G is (isomorphic to) a countable infinite subgroup of $\mathrm{Homeo}(X)$.
- An **invariant measure** of (X, G) is a Borel probability measure μ such that for every Borel set $A \subseteq X$ and $g \in G$

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Q1: Given a discrete countable group G, find examples of dynamical systems (X, G) such that $\mathcal{M}(X, G) = \emptyset$.

Amenable groups

- G is **amenable** if G has **Følner sequences:**

- **Ex:** abelian groups are amenable, $\mathbb{F}_d$ is not amenable, for $d \geq 2$.
- If G is **amenable**, then $\mathcal{M}(X, G)$ is always non-empty *:
- **Dynamical characterization of amenability:**

$$G \text{ amenable} \iff \mathcal{M}(X, G) \neq \emptyset \text{ for every } (X, G).$$

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- If G is **amenable**, then $\mathcal{M}(X, G)$ is always non-empty *: Indeed, for every $x \in X$ and $(F_n)_{n\in\mathbb{N}}$ Følner, the accumulation points of $(\frac{1}{|F_n|}\sum_{g\in F_n} \delta_{gx})_{n\in\mathbb{N}}$ are invariant measures
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Test families for amenability.

- If G is non-amenable then there exists a dynamical system (X, G), with X the Cantor set, such that $\mathcal{M}(X, G) = \emptyset$ (Giordano, de la Harpe 97).
- **Corollary**:

$$G \text{ amenable} \iff \mathcal{M}(X, G) \neq \emptyset, \text{ for every } (X, G) \text{ with } X \text{ Cantor.}$$

- The family of dynamical systems (X, G), with X Cantor, is a **test for amenability** for countable groups.
- **Q2:** Within Cantor systems, find test families for amenability for countable groups.

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Subshifts: A G-subshift is a closed set $X \subseteq \Sigma^G$ which is invariant under the shift action. The dynamical system (X, G) is also called a G-subshift.

Special class of groups: residually finite groups.

- G is **residually finite** if the intersection of all its subgroups of finite index is trivial.
- **Ex:** $\mathbb{Z}$, $\mathbb{Z}^d$, $\mathbb{F}_d$ are residually finite. Divisible groups as $\mathbb{Q}$, are not residually finite.
- **Theorem:**[†] The group G is residually finite if and only if the set of points of Σ^G which have a finite orbit with respect to the shift action, is dense in Σ^G.
- **Corollary:** If G is residually finite, then (Σ^G, G) has plenty of invariant measures supported on finite orbits (even when G is non-amenable).

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- (X, G) is **minimal** if $O_G(x) = \{gx : x \in X\}$ is dense in X, for every $x \in X$.
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Toeplitz G-subshifts: a good family of subshifts.

An element $x \in \Sigma^G$ is **Toeplitz**[‡] if for every $g \in G$ there exists a finite index subgroup Γ of G such that

$$x(g) = x(\gamma g) = \sigma^{\gamma^{-1}} x(g) \text{ for every } \gamma \in \Gamma.$$

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If there exists an aperiodic element $x \in \Sigma^G$, then G is **residually finite**.

Existence of aperiodic Toeplitz subshifts.

Proposition (C., Petite 2008; Krieger 2007): The following sentences are equivalent:

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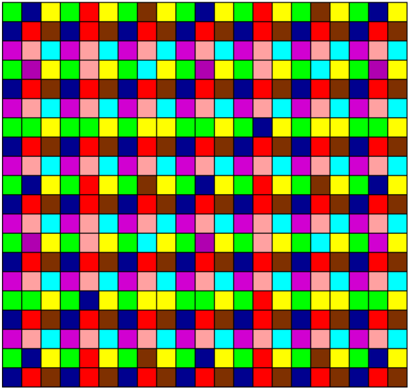
Remark: There are aperiodic Toeplitz subshifts in $\{0,1\}^{\mathbb{Z}^d}$, $\{0,1\}^{\mathbb{F}_2}$ but not in $\{0,1\}^{\mathbb{Q}}$.

Example



An aperiodic Toeplitz element in $\Sigma^{\mathbb{Z}^2}$, where Σ is an alphabet with 8 letters (colors).

Back to the main question.

Q2: Within Cantor systems, find test families for amenability for countable groups.

- G is amenable $\iff \mathcal{M}(X, G) \neq \emptyset$, for every (X, G).
- G is amenable $\iff \mathcal{M}(X, G) \neq \emptyset$, for every (X, G), with X a Cantor set (Giordano, de la Harpe 1997).
- G is amenable $\iff \mathcal{M}(X, G) \neq \emptyset$, for every (X, G), subshift (deduced from GdlH97, or Seward 2014).
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Question: Residually finite groups are characterized by the existence of non-periodic Toeplitz subshifts. Are these subshifts a test family for the amenability of residually finite groups?

Corollary (C., Gómez 2024)

Let G be a countable residually finite group. The following statements are equivalent:

- *G is non-amenable.*
- *There exists an aperiodic Toeplitz subshift (X, G) such that $\mathcal{M}(X, G) = \emptyset$.*
- *For every aperiodic minimal equicontinuous Cantor system (Z, G), there exists a Toeplitz subshifts (X, G) whose maximal equicontinuous factor is (Z, G) and such that $\mathcal{M}(X, G) = \emptyset$.*

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Remark: Todor Tsankov shows the same result using Boolean algebras tools (non-published work).

Almost 1-1 extensions

(X, G) is an **extension** of (Y, G) if the exists a continuous surjective map $\pi : X \to Y$ such that

$$\pi(gx) = g\pi(x) \text{ for every } g \in G, x \in X.$$

[§] For $G = \mathbb{Z}$, Downarowicz and Lacroix 1998

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$$\{y \in Y : |\pi^{-1}\{y\}| = 1\} \text{ is dense}$$

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Characterizations of Toeplitz subshifts(C. Petite 2008; Krieger 2007)[§]: Let $X \subseteq \Sigma^G$ be a minimal subshift.

$$X \text{ is Toeplitz} \quad \Longleftrightarrow \quad (X, G) \text{ is an almost } 1-1 \text{ extension of a minimal, equicontinuous, Cantor system}$$

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Almost 1-1 extensions of Furstenberg-Weiss type.

$$
\begin{tikzcd}
 & M(Y,G) \\
Y \arrow[ur, squiggly] \arrow[d] & \\
Z &
\end{tikzcd}
$$

Almost 1-1 extensions of Furstenberg-Weiss type.

$$
\begin{tikzcd}
 & M(Y,G) \arrow[rr, equal] & & M(X,G) \\
Y \arrow[ur, squiggly] \arrow[rr, dashed] \arrow[d] & & X \arrow[ur, squiggly] \arrow[dll, "\text{almost } 1\text{–}1"] & \\
Z & & &
\end{tikzcd}
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$$
\begin{tikzcd}
 & M(Y,G) \arrow[r, "1-1"] & M(X,G) \\
Y \arrow[ur, squiggly] \arrow[r, dashed, "\text{measurable}"] \arrow[d] & X \arrow[ur, squiggly] \arrow[dl, "\text{almost } 1-1"] & \\
Z & &
\end{tikzcd}
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(Furstenberg, Weiss 1989) For Z minimal and Y transitive. Borel measurable injective $\theta : Y \to X$ and 1-1 affine map θ^*.

Almost 1-1 extensions of Furstenberg-Weiss type.

$$
\begin{tikzcd}
 & M(Y,\mathbb{Z}) \arrow[r, "\text{homeo}"] & M(X,\mathbb{Z}) \\
Y \arrow[ru, squiggly] \arrow[r, dashed, "\text{measurable}"] \arrow[d] & X \arrow[ru, squiggly] \arrow[dl, "\text{almost } 1-1"] & \\
Z & &
\end{tikzcd}
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(Downarowicz, Lacroix 1998) For $\mathbb{Z}$-actions and Z minimal strictly ergodic (or symbolic).

Almost 1-1 extensions of Furstenberg-Weiss type.

Theorem (C., Gómez 2024)

Let G be an infinite residually finite group. Let (Z, G) be a minimal equicontinuous Cantor system having a free orbit. Let (Y, G) be a minimal extension of (Z, G) through a factor map $\pi : Y \to Z$. Then there exist a minimal extension (X, G) of (Z, G) through an almost 1-1 factor map $\tau : X \to Z$, and a Borel equivariant map $\psi : Y \to X$, satisfying the following:

1. $\tau \circ \psi = \pi$.
2. *ψ is $1 - 1$ on $\pi^{-1}(\tilde{Z})$, where $\tilde{Z} \subseteq Z$ is a full-measure set with respect to the unique invariant probability measure in (Z, G).*
3. *The map ψ induces an affine bijection ψ^* between $M(Y, G)$ and $M(X, G)$.*
4. *If in addition, Y is a Cantor set, then X is also a Cantor set and the map ψ^* is an affine homeomorphism.*

$$
\begin{tikzcd}
& M(Y,G) \arrow[r, "\text{bijection}"] & M(X,G) \\
Y \arrow[ur, squiggly] \arrow[r, dashed] \arrow[d] & X \arrow[ur, squiggly] \arrow[dl, "\text{almost } 1\text{–}1"] & \\
Z & &
\end{tikzcd}
$$

For Y Cantor,

$$
\begin{tikzcd}
& M(Y,G) \arrow[r, "\text{homeo}"] & M(X,G) \\
Y \arrow[ur, squiggly] \arrow[r, dashed] \arrow[d] & X \arrow[ur, squiggly] \arrow[dl, "\text{almost } 1\text{–}1"] & \\
Z & &
\end{tikzcd}
$$

Sketch of the proof

- Fix $y_0 \in \bigcap_{i\in\mathbb{N}} \pi^{-1}C_i$. We define $\phi : Y \to Y^G$ as follows: for every $g \in G$ and $y \in Y$,

$$\phi(y)(g) = \begin{cases} d_{i-1}^{-1}y_0 & \text{if } g \in J_i(y) \text{ and } d_{i-1} \in D_{i-1} \text{ is such that} \\ & \quad g \in t_{i,y}\Gamma_i d_{i-1} \\ g^{-1}y & \text{if } g \in \mathrm{Aper}(y). \end{cases},$$

- The function $\psi : Y \to Z \times Y^G$, $\psi(y) = (\pi(y), \phi(y))$ is measurable and equivariant. Furthermore, if $X = \overline{\{g\psi(y_0) : g \in G\}} \subseteq Z \times Y^G$.Then
  1. (X, G) is a minimal almost one-to-one extension of (Z, G) through the projection on the first coordinate $\tau : X \to Z$. In particular,
  $$\tau^{-1}\{\pi(y_0)\} = \{\psi(y_0)\}.$$
  2. $\psi(Y) \subseteq X$,
  3. The system (X, G) is Cantor if (Y, G) is.

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- ψ^* is injective.
- For each $\mu \in M(X, G)$, we have $\mu(\psi(Y)) = 1$ (ψ^* is surjective).
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Test of amenability

Corollary (C., Gómez 2024)

Let G be a non-amenable residually finite group. Then for every minimal equicontinuous Cantor system (Z, G) having a free orbit, there exists a Toeplitz G-subshift with no invariant probability measures, whose maximal equicontinuous factor is (Z, G).

Sketch of the proof

Lemma. There exists Y a Cantor system without invariant measures and a factor map $\phi : Y \to Z$

Apply previous theorem

$$\begin{tikzcd}
 & \emptyset & \emptyset \\
Y \arrow[ur, squiggly] \arrow[r, dashed] \arrow[d] & X' \arrow[ur, squiggly] \arrow[dl, "\text{alm.1-1}"] & \\
Z & &
\end{tikzcd}$$

Lemma.

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tau"] \arrow[d] \arrow[dr, "\text{alm.1-1}"] & S' \arrow[d, "\text{alm.1-1}"] \\
X' \arrow[r, "\text{alm.1-1}"'] & Z
\end{tikzcd}$$

Lemma.

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] \arrow[dd, bend left=80, "\text{alm.1-1}"] & \\
S'' & \tilde{X} \arrow[l] \arrow[u] \arrow[r, "\tau"] \arrow[d, "\text{alm.1-1}"'] & S' \arrow[dl, "\text{alm.1-1}"] \\
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$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tau"] \arrow[d] \arrow[dr, red, "\text{alm.1-1}"] & S' \arrow[d, "\text{alm.1-1}"] \\
X' \arrow[r, "\text{alm.1-1}"'] & Z
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Lemma.

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] \arrow[dd, bend left=60, "\text{alm.1-1}"] & \\
S'' & \tilde{X} \arrow[l] \arrow[u] \arrow[r, "\tau"] \arrow[d, "\text{alm.1-1}"'] & S' \arrow[dl, "\text{alm.1-1}"] \\
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Realization of non-empty sets of measures.

- If non-empty, $\mathcal{M}(X, G)$ is a **metrizable Choquet simplex:** a compact, convex, and metrizable subset K of a locally convex vector space, such that for every $x \in K$ there exists a unique probability measure ν supported on the extreme points of K, such that $\int f d\nu = f(x)$, for every continuous linear functional f.
- Example: finite dimensional simplices, Poulsen simplex (the extreme points are dense)
- **Q3:** If K is a metrizable Choquet simplex, is there a dynamical system (X, G) such that $K = \mathcal{M}(X, G)$?

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What is known about the realization of Choquet simplices?

1. Restrictions depending on G (Glasner, Weiss 97).
2. G amenable + **residually finite**, then for every metrizable Choquet simplex K there exists a **minimal aperiodic** dynamical system (X, G) such that $\mathcal{M}(X, G) = K$ (Downarowicz 1991; C., Petite 2014).
3. **Q4:** If G is amenable and non-residually finite? (something in Frej,Huczek 2018; C, Cecchi 2019).
4. **Q5:** If G is non-amenable? (something in Elek 2021; Cecchi, C, Gómez 2024).
5. The dynamical systems in 2. belong to the family of **Toeplitz subshifts**.

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The non-amenable case: realization of non-empty simplices.

Theorem (Cecchi, C., Gómez 2024)

If G is a residually finite group and $(\Gamma_n)_{n\in\mathbb{N}}$ is a decreasing sequence of normal finite index subgroups of G with trivial intersection, then for every $r \geq 1$, there exists an aperiodic Toeplitz subshift (X, G) such that

1. *(X, G) is an almost 1-1 extension of the G-odometer associated to $(\Gamma_n)_{n\in\mathbb{N}}$.*
2. *(X, G) has at least r different ergodic measures $\nu_1, \cdots, \nu_r$.*
3. *(X, G, ν_i) is conjugate in measure to the G-odometer associated to $(\Gamma_n)_{n\in\mathbb{N}}$, for every $1 \leq i \leq r$.*

Questions:

Suppose that G is non-amenable and residually finite.

- Which are the simplices that correspond to the space of invariant probability measures of Toeplitz G-subshift? (some obstructions: groups with property T.)
- Are the invariant measures of a Toeplitz G-subshifts always the limit of periodic measures? (for a class of non-amenable residually finite groups, it is not true that for arbitrary systems, any invariant measures is the limit of periodic measures).
- Is there a Toeplitz G-subshift on the alphabet $\{0, 1\}$ with no invariant measures? (control on the size of the alphabet).
- More applications of the almost 1-1 extensions of Furstenberg-Weiss type.

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Gracias!!!

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