Low complexity subshift associated to one dimensional Turing machines

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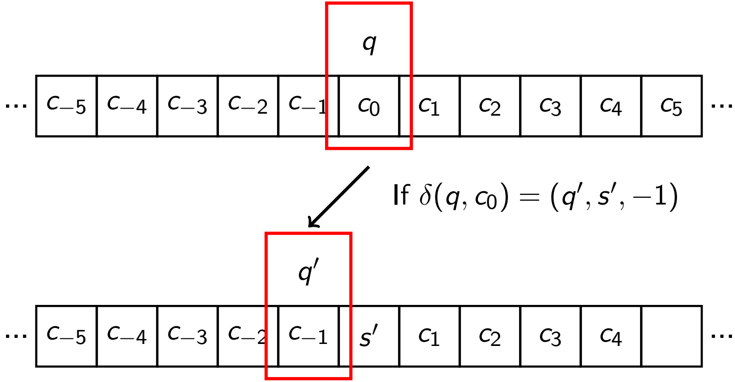
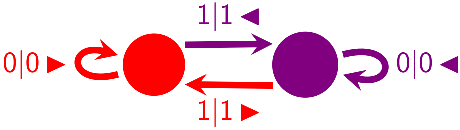
Turing Machines

$$M = (Q, \Sigma, \delta), \qquad \delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$$



Example 1: the head bounces on 1s



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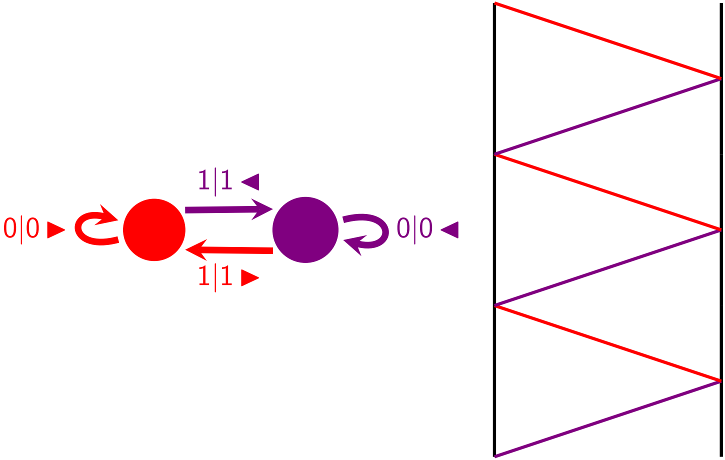
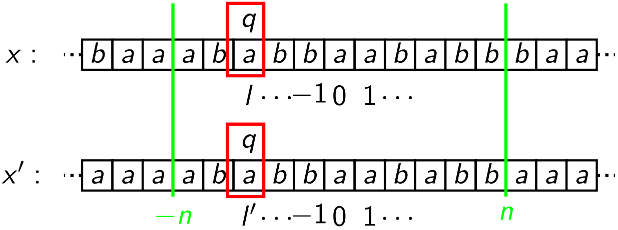


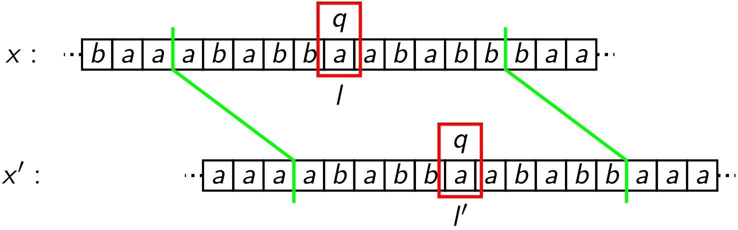
Metrics in the Moving Head Model



$$d(x, x') = 2^{-n},$$

where n is the radious of the smallest window seeing the same in x and x'

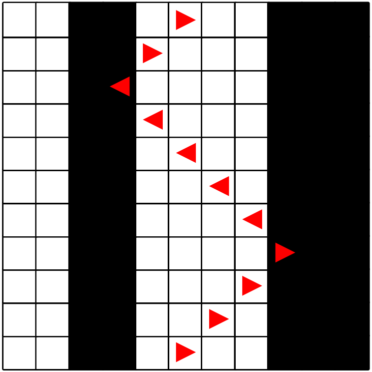
Metrics in the Moving Tape Model



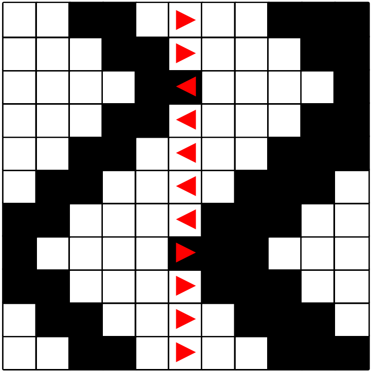
$$d(x, x') = \begin{cases} 1 & \text{If } q \neq q' \\ 2^{-n} & \text{If } n = \min\{|i - l| : x_i \neq x'_{i+l'-l}\} \wedge q = q' \\ 0 & \text{If } \sigma^{l}(x) = \sigma^{l'}(x') \wedge q = q' \end{cases}$$

Two Kůrka's dynamical models

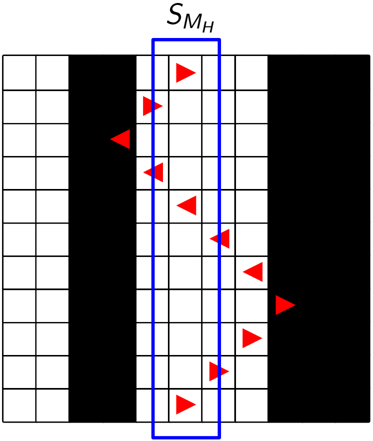
M_H

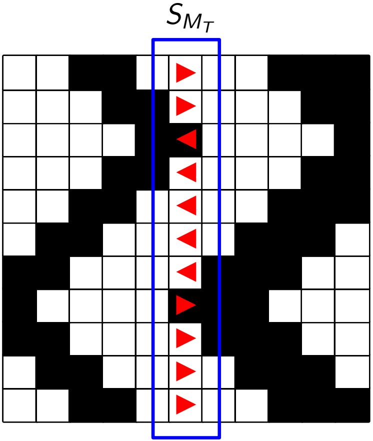


M_T



Two Kůrka's dynamical models projections





Entropy

Theorem (Jeandel, 14)

The entropy of (X_T, M_T) is computable.

$\rightsquigarrow$ The proof is based on *crossing sequences* and it uses the concept of *velocity*:
Speed of M on configuration c: $s_t(c) = \frac{\#\text{visited cells before time } t}{t}$
$\rightsquigarrow$ The maximum asymptotic speed is rational, computable and attainable on a shifting periodic point

Propagation rate

Definition

The movement bound function of a machine M is:

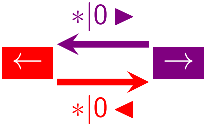
$$m(t) = \max\{s_t(c)t \ : \ c \text{ configuration}\}$$

Theorem (Guillon, Salo, 17)

Let $M = (Q, \Sigma, \delta)$ to be a Turing machine with movement bound function $m(\cdot)$. Then, exactly one of the following holds.

1. $m(t)$ is a constant $\rightsquigarrow$ M_H is bounded, entropy $= 0$
2. $m(t) = \Omega(\log t)$ and $m(t) = O(\frac{t}{\log t})$
   $\rightsquigarrow$ no shifting periodic points, entropy $= 0$
3. $m(t) = \Theta(t)$ $\rightsquigarrow$ shifting periodic points, entropy may be > 0

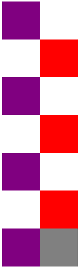
A bounded machine



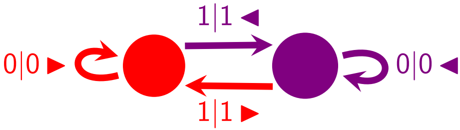
Not reversible, S_{M_T} finite

M_T pre-periodic one periodic point

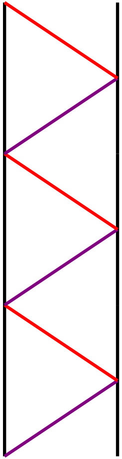
$m(t) \leq 2$, $H(S_{M_T}) = 0$



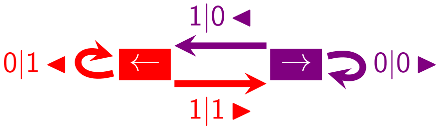
The machine that rebounds



Reversible, S_{M_T} countable
Infinite periodic points
Two shifting periodic points
$m(t) = t$, $H(S_{M_T}) = 0$



The odometer machine

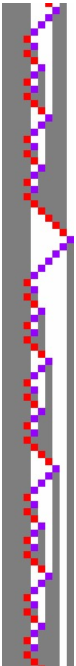


Reversible, S_{M_T} transitive

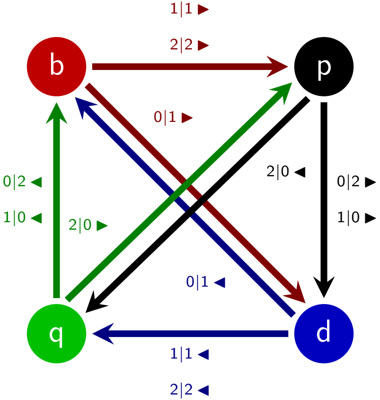
M_T not transitive

two periodic points

$m(t) = t$, $H(S_{M_T}) = 0$



SMART



Reversible, M_T minimal
M_T weak mixing
$m(t) = \Theta(log_3(t))$, $H(S_{M_T}) = 0$

Substitutions and Turing machines

Definition

A Turing machine M is called *substitutive* if there exists a substitution θ such that $S_{M_T} = X(\theta)$.

Substitutions and Turing machines

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A Turing machine M is called *substitutive* if there exists a substitution θ such that $S_{M_T} = X(\theta)$.

$\rightsquigarrow$ Most Turing machines are not substitutive.

$\rightsquigarrow$ The S_{M_T} of substitutive Turing machines have null entropy.

$\rightsquigarrow$ They have a finite number of periodic points.

$\rightsquigarrow$ Many of their properties can be obtained from θ.

The trace of SMART is substitutive with a primitive substitution

$$
\begin{array}{lcl}
b_0 & \to & q_1 b_0 d_0 b_1 \\
d_0 & \to & p_1 d_0 b_0 d_1 \\
q_2 & \to & b_2 p_2 q_2 p_0 \\
p_2 & \to & d_2 q_2 p_2 q_0 \\
q_1 & \to & b_1 p_2 \\
p_1 & \to & d_1 q_2 \\
q_0 & \to & b_0 d_2 \\
p_0 & \to & d_0 b_2 \\
d_1 & \to & p_1 d_1 q_0 \\
d_2 & \to & p_1 d_2 q_0 \\
b_1 & \to & q_1 b_1 p_0 \\
b_2 & \to & q_1 b_2 p_0
\end{array}
$$

The matrix of the substitution has eigenvalues $\{3, 1, -1\}$

The subshift has a single eigenvalue: 1

S_{M_T} is minimal and weak mixing

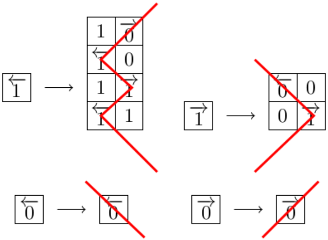
The trace of SMART is substitutive with a primitive substitution

$$\boxed{b_0} \longrightarrow \begin{array}{|c|c|}\hline b_1 & 1 \\ \hline 1 & d_0 \\ \hline b_0 & 0 \\ \hline 0 & q_1 \\ \hline\end{array} \qquad \boxed{d_0} \longrightarrow \begin{array}{|c|c|}\hline 1 & d_1 \\ \hline b_0 & 1 \\ \hline 0 & d_0 \\ \hline p_1 & 0 \\ \hline\end{array} \qquad \boxed{q_2} \longrightarrow \begin{array}{|c|c|}\hline 0 & p_0 \\ \hline q_2 & 0 \\ \hline 2 & p_2 \\ \hline b_2 & 2 \\ \hline\end{array} \qquad \boxed{p_2} \longrightarrow \begin{array}{|c|c|}\hline q_0 & 0 \\ \hline 0 & p_2 \\ \hline q_2 & 2 \\ \hline 2 & d_2 \\ \hline\end{array}$$

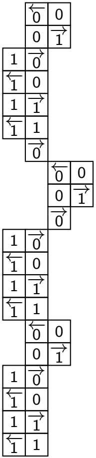
$$\boxed{q_1} \longrightarrow \begin{array}{|c|c|}\hline 1 & p_2 \\ \hline b_1 & 2 \\ \hline\end{array} \qquad \boxed{p_1} \longrightarrow \begin{array}{|c|c|}\hline q_2 & 1 \\ \hline 2 & d_1 \\ \hline\end{array} \qquad \boxed{q_0} \longrightarrow \begin{array}{|c|c|}\hline 1 & d_2 \\ \hline b_0 & 2 \\ \hline\end{array} \qquad \boxed{p_0} \longrightarrow \begin{array}{|c|c|}\hline b_2 & 1 \\ \hline 2 & d_0 \\ \hline\end{array}$$

$$\boxed{d_+} \longrightarrow \begin{array}{|c|c|}\hline q_0 & + \\ \hline 0 & d_+ \\ \hline p_1 & + \\ \hline\end{array} \qquad \boxed{b_+} \longrightarrow \begin{array}{|c|c|}\hline + & p_0 \\ \hline b_+ & 0 \\ \hline + & q_1 \\ \hline\end{array}$$

The odometer machine is also substitutive but not with a primitive substitution



Two two-sided fixed points of θ,
one in the adherence of the
shift-orbit of the other
S_{M_T} is transitive



Substitution theory $\to$ machine dynamics

$$A = \begin{array}{c c} & \begin{array}{cccc} \overleftarrow{1} & \overrightarrow{1} & \overrightarrow{0} & \overleftarrow{0} \end{array} \\ \begin{array}{c} \overleftarrow{1} \\ \overrightarrow{1} \\ \overrightarrow{0} \\ \overleftarrow{0} \end{array} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \quad v_M = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \; \mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Substitution theory $\to$ machine dynamics

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$$|\theta^n(s)| = \mathbb{1}^T A^n_s,$$

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$|\theta^n(s)| = \mathbb{1}^T A_s^n$, and the head shifting is: $v_M^T A_s^n$

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$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{1}^T A^n = \mathbb{1}^T P J^n P^{-1} = (2^{n+2} - n - 3,\ n+1,\ 1,\ 1),$

Substitution theory $\rightarrow$ machine dynamics

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$\mathbb{1}^T A^n = \mathbb{1}^T P J^n P^{-1} = (2^{n+2} - n - 3,\ n+1,\ 1,\ 1),$
$v_M^T A^n = v_M^T P J^n P^{-1} = (1+n,\ -1-n,\ 1,\ -1).$

Properties

Property

If $S_{M_T} = X(\theta)$ is minimal and the movement vector of M is v_M, then

$$m(t) = O(pol(log_\rho(t))t^{log_\rho(\alpha)}),$$

where ρ is the dominant eigenvalue of the composition matrix of θ A, and $\alpha = \max\{|\lambda| \ : \ v_M \text{ not ortogonal } F_\lambda(A) \text{ and } \lambda \neq 0\}$.

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$\rightsquigarrow$ For the Smart machine: $\rho = 3$, $\alpha = 1$, $m(t) = O(pol(log_3(t)))$

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Proof idea:

$$w = p_0\theta(p_1)\theta^2(p_2)...\theta^k(p_k s_k)...\theta^2(s_2)\theta(s_1)s_0$$

with p_i prefix of some $\theta(a)$ and s_i sufix of some $\theta(a)$, but

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with p_i prefix of some $\theta(a)$ and s_i sufix of some $\theta(a)$, but

$$\begin{aligned} m(w) &= \sum_{i=0}^{k} v_M A^i ab(p_i s_i) \sim pol(k)\alpha^k \\ t &= |w| \sim \rho^k \end{aligned}$$

Finding the substitution

Algorithm to accelerate a Turing machine (Paul's algorithm)

```
Compute the Rauzy graph of order 2 of S_{M_T};
Group together pairs with equal not 0 total movement and
equal seccond symbol;
Define a machine where:
    states are the vertices with out degree |Q|;
    transitions are the paths between them;
Call this new machine M'.
```

Example

Compute admissible words of length two in S_{M_T}.

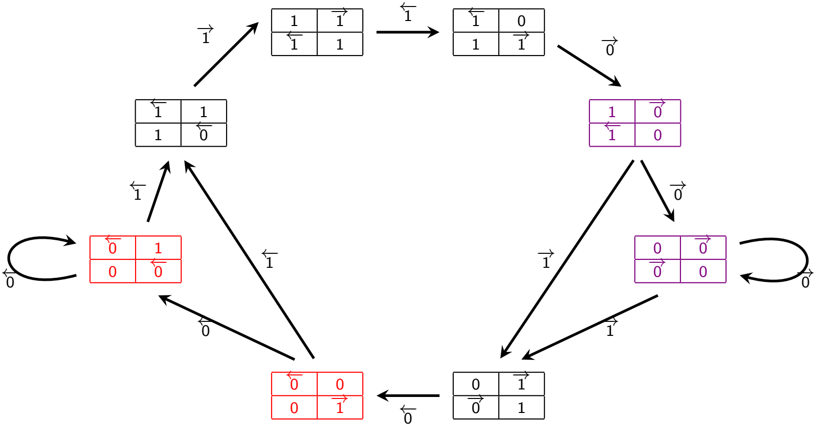
$$\begin{array}{c} - \\ - \end{array}\begin{array}{|c|c|}\hline \overleftarrow{0} & 1 \\ \hline 0 & \overleftarrow{0} \\ \hline \end{array} \qquad \begin{array}{c} + \\ - \end{array}\begin{array}{|c|c|}\hline \overleftarrow{1} & 1 \\ \hline 1 & \overleftarrow{0} \\ \hline \end{array} \qquad \begin{array}{c} + \\ + \end{array}\begin{array}{|c|c|}\hline 1 & \overrightarrow{0} \\ \hline \overleftarrow{1} & 0 \\ \hline \end{array} \qquad \begin{array}{c} - \\ + \end{array}\begin{array}{|c|c|}\hline 1 & \overrightarrow{1} \\ \hline \overleftarrow{1} & 1 \\ \hline \end{array}$$

$$\begin{array}{c} + \\ + \end{array}\begin{array}{|c|c|}\hline 0 & \overrightarrow{0} \\ \hline \overrightarrow{0} & 0 \\ \hline \end{array} \qquad \begin{array}{c} - \\ + \end{array}\begin{array}{|c|c|}\hline 0 & \overrightarrow{1} \\ \hline \overrightarrow{0} & 1 \\ \hline \end{array} \qquad \begin{array}{c} - \\ - \end{array}\begin{array}{|c|c|}\hline \overleftarrow{0} & 0 \\ \hline 0 & \overrightarrow{1} \\ \hline \end{array} \qquad \begin{array}{c} + \\ - \end{array}\begin{array}{|c|c|}\hline \overleftarrow{1} & 0 \\ \hline 1 & \overrightarrow{1} \\ \hline \end{array}$$

We already identify here the 2-length words that goes together.

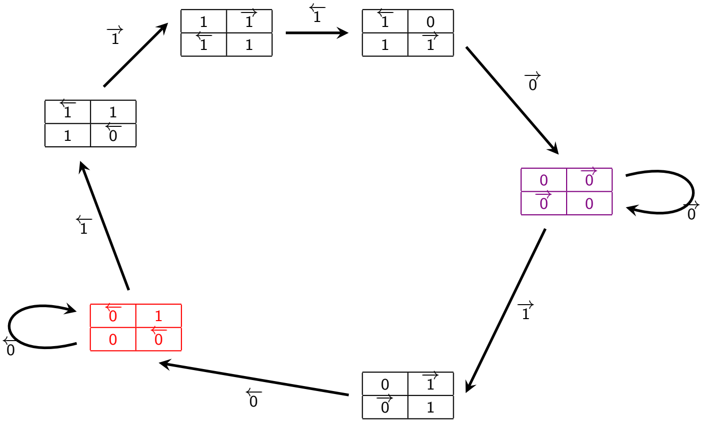
Example

Define the Rauzy graph.



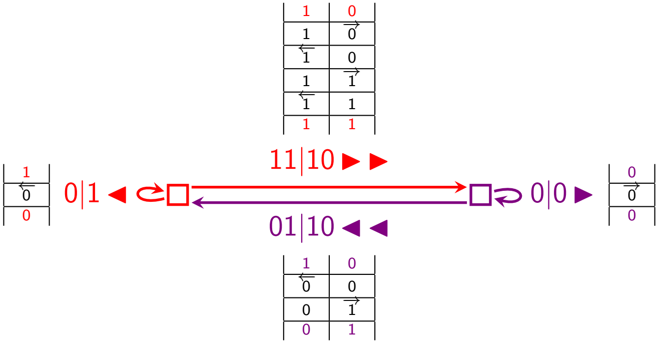
Example

Merge the *equivalent* vertices.



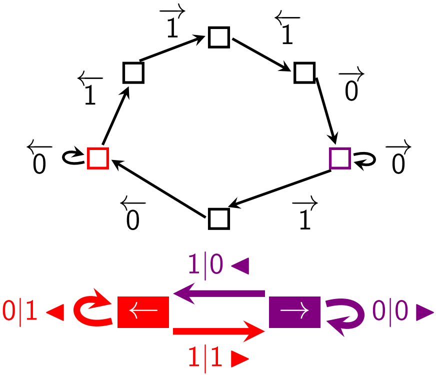
Example

Construct a new machine M'.



This machine goes twice faster in some steps, but it does the same as M.

Example



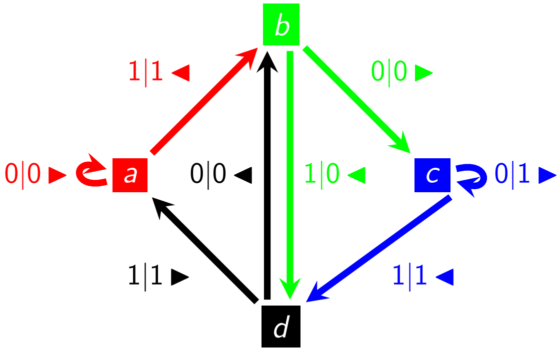
In this example, the graph of M has the same shape as the Rauzy graph with the merged vertices, and we can deduce the substitution.

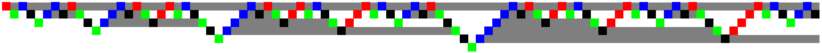
$$\overleftarrow{1} \to \overleftarrow{1}\,\overrightarrow{1}\,\overleftarrow{1}\,\overrightarrow{0} \qquad \overrightarrow{1} \to \overrightarrow{1}\,\overleftarrow{0} \qquad \overrightarrow{0} \to \overrightarrow{0} \qquad \overleftarrow{0} \to \overleftarrow{0}$$

Remarks

- If M is not equal to the merged Rauzy graph, the procedure can be repeated on M'.
- If eventually a graph similar to M is found, a substitution exists.
- The structure of M' may help to better understand M.
- Until now, all the substitutive machines we have found are kind of odometers (also Smart).

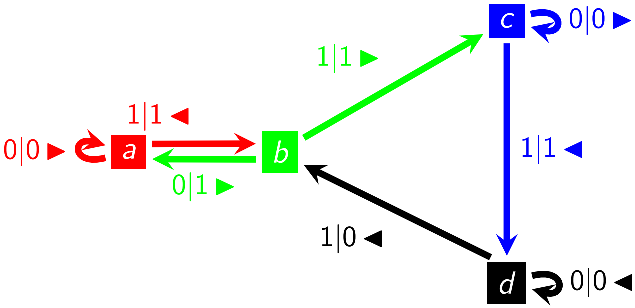
Other substitutive machines





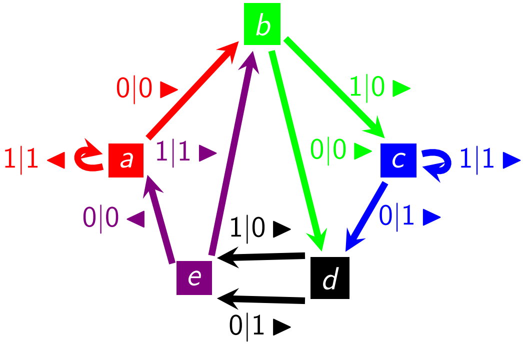
Eigenvalues: $\{2, 1, 0\}$

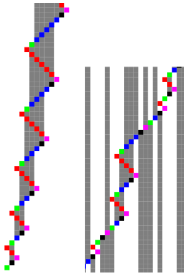
Other substitutive machines



Eigenvalues: $\{2, 1\}$

Substitutive traces over homogeneous configurations





$$
\begin{array}{lll}
a_0 \to a_1 a_0 & b_0 \to b_0 d_0 e_0 a_1 a_0 b_1 c_0 & c_0 \to c_0 \\
a_1 \to a_1 & b_1 \to b_1 c_1 & c_1 \to c_1 \\
d_0 \to d_0 & e_0 \to e_0 & \\
d_1 \to e_1 & e_1 \to b_1 &
\end{array}
$$

Eigenvalues: $\{1, 0\}$

Questions

Questions

- Which properties of Turing machines can be deduced directly from its substitution?
- Which property characterizes substitutive Turing machines?
- How to construct a Turing machine from a substitution? which substitutions corresponds to Turing machines?
- Are there Turing machine traces over homogeneous initial configurations which are not substitutive?
- Are there Turing machines with a minimal and topologicaly mixing trace shift?
- Are there Turing machines with $m(t) = \Theta(t/\log(t))$?

¡GRACIAS!