Partial rigidity rate in constant-length $\mathcal{S}$-adic subshifts

at Dyadisc 7: Brazilian, Chilean and French Interplay for Symbolic Dynamics

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Part of this project was done jointly with A. Maass and S. Donoso.

Partial rigidity

Definition $(X, \mathcal{X}, \mu, T)$ is partially rigid if there exists $\delta > 0$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that

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Special case: The system is rigid whenever $\delta = 1$, in that case

$$\lim_{k\to\infty} \mu(A \cap T^{-n_k}A) = \mu(A) \quad \text{for all} \quad A \in \mathcal{X}.$$

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- Non-superlinear complexity subshifts are *partially rigid* (Creutz 2022)

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Proposition For $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$

- If X and Y are isomorphic then $\delta_\mu(T) = \delta_\nu(S)$
- If $\pi : X \to Y$ a factor $\delta_\mu(T) \leq \delta_\nu(S)$
- $\delta_{\mu\times\mu}(T \times T) = (\delta_\mu(T))^2$

Can we compute δ_μ?

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$$\mathcal{L}^{(n)}(\boldsymbol{\sigma}) = \{w \in A_n^* \mid w \text{ appears in } \sigma_{[n,N)}(a) \text{ for } N > n, a \in A_N\}$$

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- The *$\mathcal{S}$-adic subshift given by $\boldsymbol{\sigma}$* is $X_{\boldsymbol{\sigma}}^{(0)} = X_{\boldsymbol{\sigma}}$

$\mathcal{S}$-adic subshift

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  The idea is that for $x \in X_{\boldsymbol{\sigma}}$ there is $y \in X_{\boldsymbol{\sigma}}^{(n)}$ such that

$$x = S^j(\sigma_{[0,n)}(y)) \text{ for some } j \in \mathbb{Z}$$

  An invariant measure $\mu^{(n)}$ constructed as an empirical measure in the orbit of y will induce an empirical measure in the orbit of x.

Constant-length S-adic subshift

Theorem Let $\sigma = (\sigma_n \colon A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ be a constant length directive sequence. Let μ be an ergodic measure on X_σ. Then

$$\delta_\mu = \lim_{n\to\infty} \sup_{\ell\geq 2} \left\{ \sum_{w\colon |w|=\ell,\ w_1=w_\ell} \mu^{(n)}(w) \right\}.$$

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Moreover, if $(\ell_n)_{n\in\mathbb{N}}$ is a sequence of integers (posibly constant), with $\ell_n \geq 2$ for all $n \in \mathbb{N}$, such that

$$\delta_\mu = \lim_{n\to\infty} \left(\sum_{w\colon |w|=\ell_n,\ w_1=w_{\ell_n}} \mu^{(n)}(w) \right),$$

then the partial rigidity sequence is $((\ell_n - 1)\cdot|\sigma_{[0,n)}|)_{n\in\mathbb{N}}$.

Corollary Let $\sigma : A^* \to A^*$ be a constant-length substitution and μ the unique invariant measure of X_σ.

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Remark: Fixing $u \in A^*$, $j > |u|$,

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$\downarrow \sigma$

011010011001011010010**11001101001**10010110011010010110100110010110

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Values

Proposition For the Thue-Morse substitution subshift:

$$\delta_\mu = \frac{2}{3}$$

with partial rigidity sequence $(3 \cdot 2^n)_{n \in \mathbb{N}}$

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Proposition For $L \geq 6$, let $\zeta_L \colon \{a, b\}^* \to \{a, b\}^*$ be the substitution given by

$$\zeta_L(a) = ab^{L-1}$$
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then for the invariant measure of the substitution subshift X_{ζ_L}

$$\delta_\mu = \frac{L-1}{L+1}$$

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Corollary For every $\delta \in [0, 1]$ there is a measure preserving system $(X, \mathcal{X}, \mu, T)$ such that $\delta = \delta_\mu(T)$.

System with $0 < \delta_{\mu_0} < \ldots < \delta_{\mu_{d-1}} < 1$

Recall:

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Partial rigidity rates

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- if we know $\delta_{\mu^{(\infty)}}$???? we know δ_μ.

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Define $\sigma = \Gamma(\tau_0, \ldots, \tau_{d-1})\colon \Lambda_d^* \to \Lambda_d^*$ be given by

$$\sigma(a_i) = \kappa(\tau_i(a_i))$$
$$\sigma(b_i) = \kappa(\tau_i(b_i))$$

for all $i \in \{0, \ldots, d-1\}$, the *glued substitution*.

Partial rigidity rates

Example: $d = 2$, $\tau_0 \colon \{a_0, b_0\}^* \to \{a_0, b_0\}^*$ and $\tau_1 \colon \{a_1, b_1\}^* \to \{a_1, b_1\}^*$:

$$\begin{aligned} \tau_0(a_0) &= a_0 b_0 b_0 a_0 & \tau_0(b_0) &= b_0 a_0 a_0 b_0, \\ \tau_1(a_1) &= a_1 a_1 a_1 b_1 & \tau_1(b_1) &= b_1 b_1 b_1 a_1. \end{aligned}$$

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Partial rigidity rates

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And $\sigma' = \Gamma(\tau_0^2, \tau_1^2)\colon \Lambda_2^* \to \Lambda_2^*$

$$\begin{aligned} \sigma'(a_0) &= a_0 b_0 b_0 a_0 b_0 a_0 a_0 b_0 b_0 a_0 a_0 b_0 a_0 b_0 b_0 a_1 \\ \sigma'(b_0) &= b_0 a_0 a_0 b_0 a_0 b_0 b_0 a_0 a_0 b_0 b_0 a_0 b_0 a_0 a_0 b_1 \\ \sigma'(a_1) &= a_1 a_1 a_1 b_1 a_1 a_1 a_1 b_1 a_1 a_1 a_1 b_1 b_1 b_1 b_1 a_0 \\ \sigma'(b_1) &= b_1 b_1 b_1 a_1 b_1 b_1 b_1 a_1 b_1 b_1 b_1 a_1 a_1 a_1 a_1 b_0 \end{aligned}$$

Partial rigidity rates

Theorem $\tau_i \colon \{a_i, b_i\}^* \to \{a_i, b_i\}^*$ for $i = 0, \ldots, d-1$,

$$|\tau_i| = \ell \quad \text{for some } \ell \geq 2$$

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Let $\boldsymbol{\sigma} = (\sigma_n \colon \Lambda_d^* \to \Lambda_d^*)_{n \in \mathbb{N}}$ given by

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Then the $\mathcal{S}$-adic subshift (X_σ, S) is minimal and has d ergodic measures $\mu_0, \dots, \mu_{d-1}$ such that for every $i \in \{0, \dots, d-1\}$

$$\lim_{n \to \infty} \mu_i^{(n)}(w) = \nu_i(w) \quad \text{for all } w \in \{a_i, b_i\}^*$$

where ν_i is the unique invariant measure of X_{τ_i}.

Partial rigidity rates

Corollary Under the same assumptions,

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Can we achieve equality?

Partial rigidity rates

Recall For $L \geq 6$, $i = 0, \ldots, d-1$,

$$\zeta_{L^{2^i+1}}(a) = ab^{L^{2^i+1}-1}$$
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Theorem $\sigma = (\Gamma(\tau_0^{n+1}, \dots, \tau_{d-1}^{n+1})\colon \Lambda_d^* \to \Lambda_d^*)_{n\in\mathbb{N}}$. Then,

$$\delta_{\mu_i} = \frac{L^{2^{i+1}} - 1}{L^{2^{i+1}} + 1}$$

Moreover, the partial rigidity sequence associated to δ_{μ_i} is $(|\sigma_{[0,n)}|)_{n\in\mathbb{N}}$ (in particular they are equal)

¡Gracias!