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Distribution of integers with digit restrictions in arithmetic progressions

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December 12, 2024

Introduction

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Motivation: Study the relation between n and $S_g(n)$.

Distribution of integers in congruence classes

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Let $A \subseteq \mathbb{N}$ and $a, a' \in \mathbb{N}$. For $(b, b') \in \mathbb{Z}_a \times \mathbb{Z}_{a'}$, our aim is to understand the behaviour of

$$L(b, b') := \lim_{N \to \infty} \frac{|\{n \in A \cap [0, N) : \ n \equiv b \bmod a, \ S_g(n) \equiv b' \bmod a'\}|}{|A \cap [0, N)|}.$$

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Definition

Let $a, a' \in \mathbb{N}$. The set $A \subseteq \mathbb{N}$ is said to be uniformly distributed $\bmod (a, a')$ if $L(b, b') = 1/aa'$ for all $(b, b') \in \mathbb{Z}_a \times \mathbb{Z}_{a'}$.

Gelfond's theorem

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Let $a, a' \in \mathbb{N}$. If $\gcd(g-1, a') = 1$, $\mathbb{N}$ is uniformly distributed $\mod (a, a')$.

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The hypothesis avoids the obstruction coming from the fact that $S_g(n) \equiv n \bmod g-1$.

Generalisation of Gelfond's theorem

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Theorem (S. 2024+)

$\mathbb{N}$ is uniformly distributed $\mod (a, a')$ if and only if there exists $n \in \mathbb{N}$ such that $n \equiv 0 \mod a$ and $S_g(n) \equiv 1 \mod a'$.

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Remark: From Gelfond's theorem, there are infinitely many of those n, but it is possible to provide an elementary construction using $\gcd(g-1, a') = 1$.

Integers with missing digits

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For a set of digits $\mathcal{D} \subseteq \{0, \cdots, g-1\}$, we define the missing-digits set

$$\mathcal{C}_{g,\mathcal{D}} := \left\{ \sum_{i=0}^{m} d_i g^i : \ m \in \mathbb{N}, \ d_i \in \mathcal{D} \right\}.$$

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We say that $\mathcal{C}_{g,\mathcal{D}}$ is uniformly distributed $\bmod a$ if for every $b \in \mathbb{Z}_a$,

$$L(b) := \lim_{N \to \infty} \frac{|\{n \in \mathcal{C}_{g,\mathcal{D}} \cap [0, N) : \ n \equiv b \bmod a\}|}{|\mathcal{C}_{g,\mathcal{D}} \cap [0, N)|} = 1/a.$$

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Theorem (Erdős-Mauduit-Sárközy, 1998)

Assume that $0 \in \mathcal{D}$, $\gcd(g(g-1), a) = 1$ and $\gcd\{d \in \mathcal{D}\} = 1$. Then, $\mathcal{C}_{g,\mathcal{D}}$ is uniformly distributed $\bmod a$.

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- Proof based on the circle method.
- Further generalisations were provided by Konyagin (2000) and Aloui (2015).

Equivalent condition to uniform distribution

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Theorem (S. 2024+)

Let $g \geq 2$ be an integer, and let $\mathcal{D} = \{d_1, \cdots, d_t\}$ be a set of digits, where d_1 is the smallest digit. Suppose $\gcd(g, a) = 1$ and denote

$$\delta := \gcd(a, d_2 - d_1, \cdots, d_t - d_1).$$

Then,

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Observation: A similar version also holds when considering the function sum-of-digits.

Motivation of the proof

For any $w \in \mathcal{D}^m$, let's define

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$$(dxy)_g = d + g(x_0 + x_1 g + \cdots + x_{p-1} g^{p-1}) + g^{p+1}(y_0 + y_1 g + \cdots + y_{p-1} g^{p-1}).$$

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2. For each $i \in \{1, \cdots, p\}$, we can construct Markov chains $X^i = (X^i_n)_{n\in\mathbb{N}}$ with state space $\mathbb{Z}_a$ such that

$$\mathbb{P}(X^i_n = b) = \frac{|\{w \in \mathcal{D}^{i+np} : \ (w)_g \equiv b \mod a\}|}{|\mathcal{D}^{i+np}|}.$$

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3. If $\gcd(a, d_2 - d_1, \cdots, d_t - d_1) = 1$, each Markov chain is aperiodic and irreducible. Then, every X^i converges to its invariant distribution, and we can show that

$$\lim_{N\to\infty} \frac{|\{n \in \mathcal{C}_{g,\mathcal{D}} \cap [0, g^N) : \ n \equiv b \mod a\}|}{|\mathcal{C}_{g,\mathcal{D}} \cap [0, g^N)|} = \frac{1}{a}$$

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4. Exploiting the "nice" structure of these sets, we can obtain uniform distribution.

If $\gcd(a, d_2 - d_1, \cdots, d_t - d_1) \neq 1$, we can find $b \in \mathbb{Z}_a$ and $j \in \{1, \cdots, p\}$ such that

$$\mathbb{P}(X_n^j = b) = 0$$

for all $n \in \mathbb{N}$.

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for all $n \in \mathbb{N}$. In particular, the limit $L(b)$ is either zero, or does not exists.

More general digit restrictions

Let $g = 10$, and consider the set of integers

$$A := \left\{ \sum_{j=0}^{m} w_i 10^i : \ w_i \in \{0, \cdots, 9\}, \ w_{i+1} \in \{w_i - 1, w_i, w_i + 1\} \bmod 10 \right\}.$$

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Let $a, a' \in \mathbb{N}$ such that $\gcd(10, a) = \gcd(a, a') = 1$. Therefore,

$$\lim_{N \to \infty} \frac{|\{n \in A \cap [0, N) : \ n \equiv b \bmod a, \ S_{10}(n) \equiv b' \bmod a'\})|}{|A \cap [0, N)|} = \frac{1}{aa'}$$

for every $(b, b') \in \mathbb{Z}_a \times \mathbb{Z}_{a'}$.

When there are much more general restrictions, is it possible to say anything?

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Yes, by understanding these sets as fractals in the integers!

Transversality in fractal sets

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Furstenberg's conjecture (1967): Let $A \subset [0,1]$ be closed and invariant under $\times 2 \bmod 1$, and let $B \subset [0,1]$ be closed and invariant under $\times 3 \bmod 1$. Then, the fractal sets A and B are *transverse* (they have a "lack of common structure").

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This conjecture was solved independently by Shmerkin and Wu.

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This conjecture was solved independently by Shmerkin and Wu. Motivated by an integer version of this conjecture, Glasscock, Moreira and Richter introduced the notion of multiplicatively invariant sets of integers.

Multiplicatively invariant sets of integers

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Definition

Let $g \geq 2$ and let $\Sigma \subseteq \{0, \cdots, g-1\}^{\mathbb{N}_0}$ be a subshift. We define the set generated by Σ as

$$A_\Sigma := \Big\{ (w)_g : \ w \in \mathcal{L}(\Sigma) \Big\}.$$

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Definition

Let $A \subseteq \mathbb{N}_0$ be non-empty. We define the *lower mass dimension* and the *upper mass dimension* of A as:

$$\underline{\dim}_{\mathrm{M}}(A) := \liminf_{N\to\infty} \frac{\log |A \cap [0,N)|}{\log N}, \quad \overline{\dim}_{\mathrm{M}}(A) := \limsup_{N\to\infty} \frac{\log |A \cap [0,N)|}{\log N}.$$

If $\underline{\dim}_{\mathrm{M}}(A) = \overline{\dim}_{\mathrm{M}}(A)$, we denote this value by $\dim_{\mathrm{M}}(A)$ and say that the *mass dimension* of A exists.

Open question posed by Glasscock, Moreira and Richter

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Let A_Σ be a multiplicatively invariant set and P be any arithmetic progression. Is it true that $\dim_{\mathsf{M}}(A_\Sigma \cap P)$ is either zero or $\dim_{\mathsf{M}}(A_\Sigma)$?

Partial answer and conjecture

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Theorem (S. 2024+)

Let $g \geq 2$ and let $\Sigma \subset \{0, \cdots, g-1\}^{\mathbb{N}_0}$ be a sofic and transitive subshift. Then, for any arithmetic progression, $\dim_M(A_\Sigma \cap P)$ is either zero or $\dim_M(A_\Sigma)$. This dichotomy is not always true is Σ is only sofic, or only transitive.

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A subshift Σ is said to be *entropy minimal* if for any other subshift $\Sigma' \subsetneq \Sigma$, $h_{\text{top}}(\Sigma) > h_{\text{top}}(\Sigma')$.

Conjecture

Σ is entropy minimal if and only if for every arithmetic progression P, $\dim_{\mathsf{M}}(A_\Sigma \cap P)$ is either zero or $\dim_{\mathsf{M}}(A_\Sigma)$.

Thanks for your attention!

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