Generalised Fibonacci shifts in the Lorentz attractor

Víctor F. Sirvent

Joint work with B. San Martín

Universidad Católica del Norte – Atofagasta

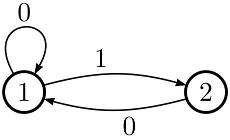
Dyadisc 7 Congress – Valparaíso

December 2024

Fibonacci shift:
$(\mathcal{F}, \sigma)$, where $\sigma(a_0 a_1 a_2 \ldots) = a_1 a_2 \ldots$ is the shift map and

$$\mathcal{F} := \{\mathbf{a} = a_0 a_1 \ldots \mid a_i \in \{0,1\},\ a_i \cdot a_{i+1} = 0,\ i \geq 0\}$$

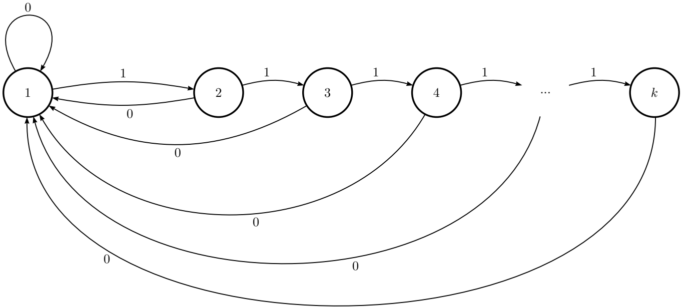
Edge shift



***k*-bonacci shift:**
$k \geq 2$, $(X(k), \sigma)$, where $\sigma(a_0 a_1 a_2 \dots) = a_1 a_2 \dots$.

$$X(k) := \{\mathbf{a} = a_0 a_1 \dots \mid a_i \in \{0,1\},\ a_i a_{i+1} \cdots a_{i+k-1} = 0,\ i \geq 0\}$$

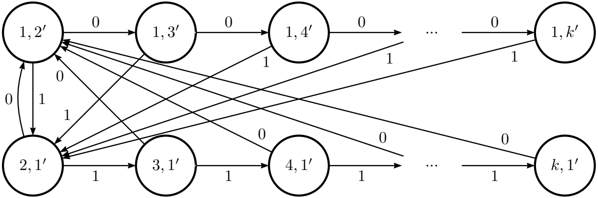
In particular $X(2) = \mathcal{F}$.
Edge shift



We would like to have the maximal sub-shift invariant under the involution $0 \leftrightarrow 1$.

$$Y(k) := \left\{ \mathbf{a} = a_0 a_1 \ldots \mid a_i \in \{0,1\}, \prod_{j=0}^{k-1} a_{i+j} = 0, \sum_{j=0}^{k-1} a_{i+j} > 0, \, , i \geq 0 \right\}$$

Edge shift



Product automaton

$$Y(k) := \left\{ \mathbf{a} = a_0 a_1 \ldots \mid a_i \in \{0,1\}, \prod_{j=0}^{k-1} a_{i+j} = 0, \sum_{j=0}^{k-1} a_{i+j} > 0, \ , i \geq 0 \right\}$$

Theorem (S. – 1996)

The following diagram commutes

$$\begin{array}{ccc} Y(k+1) & \xrightarrow{\sigma} & Y(k+1) \\ \downarrow \phi & & \downarrow \phi \\ X(k) & \xrightarrow{\sigma} & X(k) \end{array}$$

where ϕ is a 2 to 1 map.

$$Y(k) := \left\{ \mathbf{a} = a_0 a_1 \ldots \mid a_i \in \{0,1\}, \prod_{j=0}^{k-1} a_{i+j} = 0, \sum_{j=0}^{k-1} a_{i+j} > 0, \ , i \geq 0 \right\}$$

Theorem (S. – 1996)

The following diagram commutes

$$\begin{array}{ccc} Y(k+1) & \xrightarrow{\sigma} & Y(k+1) \\ \downarrow \phi & & \downarrow \phi \\ X(k) & \xrightarrow{\sigma} & X(k) \end{array}$$

where ϕ is a 2 to 1 map.

Similar result of the adic systems.

It has consequences in Rauzy fractals (S. 2011) and Lorenz attractor.

This invariance has implication in the symmetry of the Rauzy fractals.

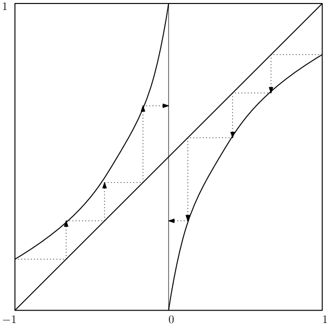
(More general setting)

1-D Lorenz map

Let $f : [-1,1] \setminus \{0\} \to [-1,1]$ be a smooth mat such that:

- f has a discontinuity at 0
- f is increasing in $[-1,0)$ with $-1 = f(0^+) < f(-1) < 0$ and increasing in $(0,1]$ with $0 < f(1) < f(0^-) = 1$.
- There exists $\lambda > 1$ such that $f'(x) \geq \lambda$, $\forall x \neq 0$ and $\lim_{x \to 0} f'(x) = \infty$.

If $\lambda \geq \sqrt{2}$ then f is transitive.

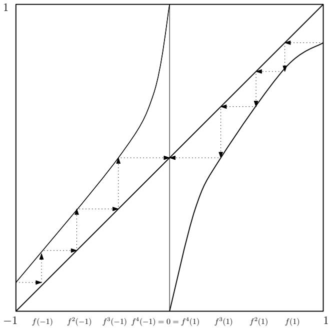


We deal with special case:

- symmetric: $-f(x) = f(-x)$
- homoclinic: there exists k such that $f^k(-1) = 0$
- some order in the orbit of -1

Particular case: k-**bonacci**:

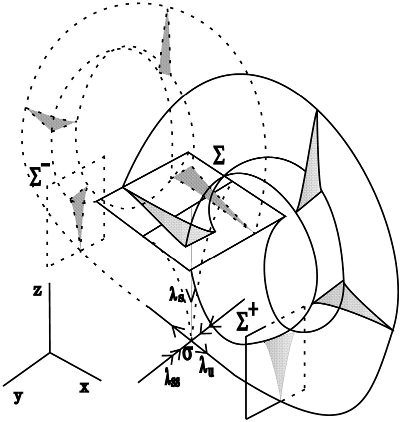
$$-1 < f(-1) < f^2(-1) < \cdots < f^{k-1}(-1) < f^k(-1) = 0$$



Geometric Lorenz Attractor (Guckenheimer – Williams)

- Let $\mathbf{X}$ be the field of the geometric Lorenz attractor.
- $\mathbf{0}$ is a singularity. $D_{\mathbf{0}}\mathbf{X}$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$: $\lambda_2 < \lambda_3 < 0 < \lambda_1$
- The **homoclinic case** is when both branches of the unstable manifold of $\mathbf{0}$ meet its stable manifold.
- Let Λ be the geometric Lorenz attractor
- Let $\mathbf{F}$ be the 1st return map of $\mathbf{X}$ for some cross section
- In suitable coordinates we have $\mathbf{F}(x, y) = (f(x), g(x, y))$

Geometric Lorenz Attractor



Theorem (Lizana – Mora: 2008)

Let Λ be the geometric Lorenz attractor for the symmetric homoclinic case.
Then there exists $\gamma \in (0,1)$ such that

$$dim_H(\Lambda) \geq 2 + \frac{\log \rho(A)}{\log(\gamma^{-1})} > 2$$

where $\rho(A)$ is the spectral radius of the matrix A, a $\{0,1\}$- matrix which describes the geometric distribution of Λ in the direction given by the eigenvector associated with λ_3.

Theorem (Mora: 2012)

$$\rho(A) \geq \frac{1+\sqrt{5}}{2}$$

It turns out the matrix A is associated with the incidence matrix of the shifts $Y(k)$.

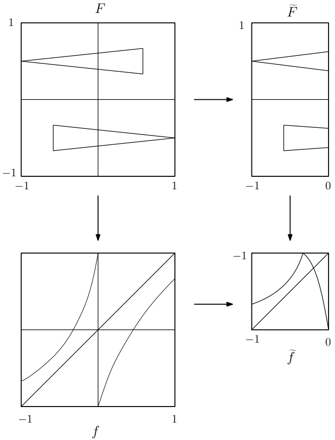
Corollary (of the proof of the Theorem)

Let f be the 1-d Lorenz map satisfaying the k-bonacci condition. Then the following diagram commutes

$$\begin{array}{ccc} Y(k+1) & \xrightarrow{\sigma} & Y(k+1) \\ \downarrow \Theta & & \downarrow \Theta \\ I & \xrightarrow{f} & I \end{array}$$

where Θ is a continuous and onto map.

In the symmetric case, we consider the quotient map $\widetilde{f} : [-1, 0] \to [-1, 0]$ defined by the equivalence relation: $x \sim -x$



Theorem (San Martín – S.)

Let f be a symmetric 1-d Lorenz map such that:

$$-1 < f(-1) < f^2(-1) < \cdots < f^{k-1}(-1) < f^k(-1) < 0,$$

with $k \geq 2$. Let $\widetilde{f} : [-1,0] \to [-1,0]$ be quotient map defined by the relation $x \sim -x$. Then the following diagram commutes:

$$\begin{array}{ccc} X(k) & \xrightarrow{\sigma} & X(k) \\ \downarrow \Psi & & \downarrow \Psi \\ [-1,0] & \xrightarrow{\widetilde{f}} & [-1,0] \end{array}$$

where Ψ is a continuous and onto map. Moreover Ψ fails to be injective in a countable set where is 2 to 1.

Corollary

The topological entropy of f and $\widetilde{f}$ is $\log(\lambda_k)$ where λ_k is the only real root greater than 1 of

$$x^k - x^{k-1} - \cdots - x - 1$$

We have the situation where this "cube" commutes.

$$\begin{tikzcd}
 & Y(k+1) \arrow[rr, "\sigma"] \arrow[dl, "\Xi"'] \arrow[dd, "\Theta"] & & Y(k+1) \arrow[dl, "\Xi"'] \arrow[dd, "\Theta"] \\
X(k) \arrow[rr, "\sigma"] \arrow[dd, "\Psi"] & & X(k) \arrow[dd, "\Psi"] & \\
 & I \arrow[rr, "f"] \arrow[dl, "\sim"] & & I \arrow[dl, "\sim"] \\
\tilde{I} \arrow[rr, "\widetilde{f}"] & & \tilde{I} &
\end{tikzcd}$$

λ_k be the larger root of $x^k - x^{k-1} - \cdots - x - 1$:

$$\frac{1+\sqrt{5}}{2} = \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots < 2$$

Corollary

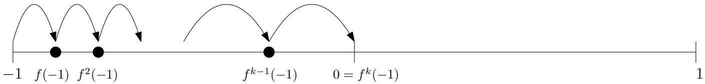
Let

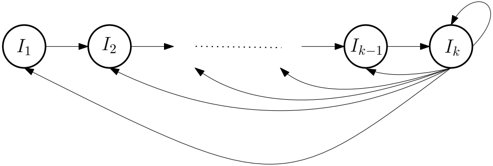
$$M := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and (Σ_M, σ) *be the sub-shift of finite type defined in* $\{1, \ldots, k\}^{\mathbb{N}}$ *by* M. *Then* (Σ_M, σ) *is semi-conjugate to* $([-1, 0], \widetilde{f})$.

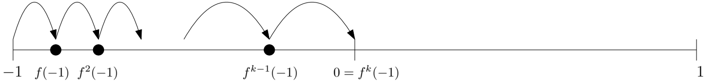
Idea: $\Sigma_M \leftrightarrow X(k)$.

Idea: $\Sigma_M \leftrightarrow X(k)$:



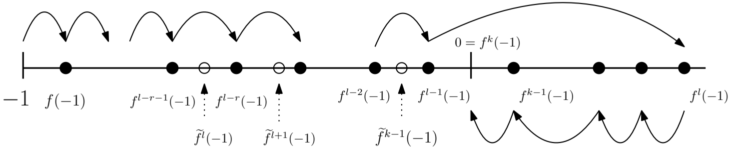


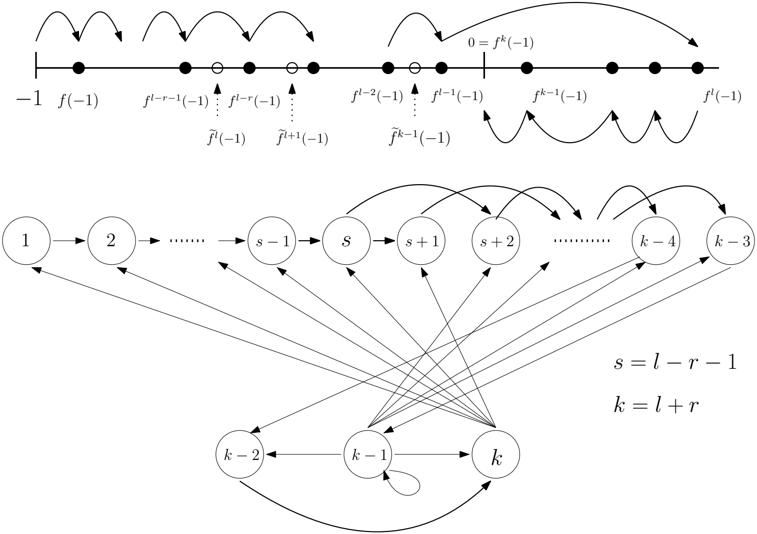
- k-bonacci case:



- (ℓ, r) case: $\ell + r = k$:

$$-1 < f(-1) < f^2(-1) < \cdots < f^{\ell-1}(-1) < 0 < f^{\ell+r-1}(-1) < \cdots < f^{\ell+1}(-1) < f^{\ell}(-1) < 1$$




$0 = f^k(-1)$
-1
$f(-1)$
$f^{l-r-1}(-1)$
$f^{l-r}(-1)$
$f^{l-2}(-1)$
$f^{l-1}(-1)$
$f^{k-1}(-1)$
$f^l(-1)$
$\tilde{f}^l(-1)$
$\tilde{f}^{l+1}(-1)$
$\tilde{f}^{k-1}(-1)$
1
2
$s-1$
s
$s+1$
$s+2$
$k-4$
$k-3$
$k-2$
$k-1$
k

$$s = l - r - 1$$

$$k = l + r$$

Theorem (San Martín – S.)

Let f be a symmetric 1-d Lorenz map of type (ℓ, r) with $\ell + r = k > 3$, $\ell > r$. Then there exists a matrix $M = M(\ell, r)$ whose characteristic polynomial is

$$x^k - x^{k-1} - \cdots - x^r + x^{r-1} + \cdots + x + 1. \tag{1}$$

such that the shift (Σ_M, σ) is semiconjugate to $([-1, 0], \widetilde{f})$.

Corollary

The topological entropy of the Lorenz map f is $\log \rho$, where ρ is the dominant root of the polynomial (1). Moreover:

$$\log \lambda_{\ell-1} \leq h(f) < \log \lambda_\ell,$$

with equality only in the case $r = \ell - 1$.

Open Problems

1. Extend this analysis to a more complicated combinatorics of the orbit. (Work in progress)

Open Problems

1. Extend this analysis to a more complicated combinatorics of the orbit. (Work in progress)
2. Let $p_{\ell,r}(x)$ be the characteristic polynomial of the matrix $M(\ell, r)$.
   - Conjecture: The dominant root of $p_{k-1,1}$ is a Salem number.
   - For k odd: -1 is a root: Is $p_{k-1,1}(x)/(x+1)$ irreducible?
   - For k even: $p_{k-1,1}$ is not always irreducible. For $k = 14, 20, 26$ the cyclotomic polynomial $1 - x + x^2$ is a factor
   - We conjecture: For $r \geq 2$ and $\ell - r \geq 2$, the dominant root of $p_{\ell,r}(x)$ is not Pisot nor Salem. We get several roots of modulus greater than 1.

1. Extend this analysis to a more complicated combinatorics of the orbit. (Work in progress)
2. Let $p_{\ell,r}(x)$ be the characteristic polynomial of the matrix $M(\ell,r)$.
   - Conjecture: The dominant root of $p_{k-1,1}$ is a Salem number.
   - For k odd: -1 is a root: Is $p_{k-1,1}(x)/(x+1)$ irreducible?
   - For k even: $p_{k-1,1}$ is not always irreducible. For $k = 14, 20, 26$ the cyclotomic polynomial $1 - x + x^2$ is a factor
   - We conjecture: For $r \geq 2$ and $\ell - r \geq 2$, the dominant root of $p_{\ell,r}(x)$ is not Pisot nor Salem. We get several roots of modulus greater than 1.
3. Is there any other relation between Rauzy fractals and the Lorentz attractor? (**Bold conjecture**)

Open Problems

1. Extend this analysis to a more complicated combinatorics of the orbit. (Work in progress)
2. Let $p_{\ell,r}(x)$ be the characteristic polynomial of the matrix $M(\ell, r)$.
   - Conjecture: The dominant root of $p_{k-1,1}$ is a Salem number.
   - For k odd: -1 is a root: Is $p_{k-1,1}(x)/(x+1)$ irreducible?
   - For k even: $p_{k-1,1}$ is not always irreducible. For $k = 14, 20, 26$ the cyclotomic polynomial $1 - x + x^2$ is a factor
   - We conjecture: For $r \geq 2$ and $\ell - r \geq 2$, the dominant root of $p_{\ell,r}(x)$ is not Pisot nor Salem. We get several roots of modulus greater than 1.
3. Is there any other relation between Rauzy fractals and the Lorentz attractor? (**Bold conjecture**)
4. Are the adic systems associated to these shifts present in some way in the Lorenz attractor? (**Bold conjecture**)

Bibliography

1. V.S. Afraimovich, V.V. Bykov and L.P. Shil'nikov , On the appearance and structure of the Lorenz attractor, *Dokl. Acad. Sci. USSR*, **234** (1977), 150–212.
2. J. Guckenheimer , A Strange, Strange Attractor, in *The Hopf Bifurcation and its Applications (New York: Springer)*,**Vol. 19** (1976), 368-381.
3. J. Guckenheimer and R.F. Williams, Structural stability of Lorenz attractors, *Publ. Math. IHES.*, **50** (1979), 59–72.
4. R. Labarca and C.G. Moreira, Essential dynamics for Lorenz maps on the real line and the lexicographical world. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **23** (2006), 683–694.
5. C. Lizana and L. Mora, Lower bounds for the Hausdorff dimension of the geometric Lorenz attractor: The homoclinic case. *Discrete Contin. Dyn. Syst.*, **22** (2008), 699–709.
6. L. Mora, A note on the lower bounds for the Hausdorff dimension of the geometric Lorenz attractor. *Bol. Asoc. Mat. Venez.*, **19** (2012), 141–145.
7. B. San Martín and V.F. Sirvent, Generalized Fibonacci shifts in the Lorenz Attractor, *Chaos, Solitons and Fractals*, **169** (2023) 113239.
8. V.F. Sirvent, Relationships between the dynamical systems associated to the Rauzy substitutions. *Theoret. Comput. Sci.*, **164** (1996), 41–57.
9. V.F. Sirvent, Symmetries on k-bonacci adic systems. *Integers*, **11B** (2011), Paper No. A15, 18 pp.