Effective equidistribution of orbits of semisimple groups on congruence quotients

Andreas Wieser

ETH Zurich and Northwestern University

Let $X = \Gamma \backslash G$ be a homogeneous space where G is a Lie group and $\Gamma < G$ is a lattice. We are interested in this talk in the dynamics of *semisimple* subgroups of G.

Theorem (Ratner's measure classification)

Let $H < G$ be a closed subgroup generated by unipotents. Then any H-invariant and ergodic probability measure μ on X is homogeneous i.e. there is a finite volume orbit xL of a closed subgroup $H \subset L \subset G$ such that μ is the L-invariant probability measure on X.

A theorem of Mozes-Shah

Relying on Ratner's theorem plus the linearization technique:

Theorem (Mozes-Shah '95)

Let μ_i be a sequence of probability measures on X such that μ_i is invariant and ergodic under some one-parameter unipotent subgroup of G. Suppose that μ_i converges to a probability measure μ. Then μ is homogeneous.

'Equidistribution unless there is an algebraic obstruction'

Goal for today: Discuss a polynomially effective versions for equidistribution of orbits of semisimple subgroups of G.

Uniform spectral gap

Notation (from now on)

- $\mathbb{G}$ semisimple $\mathbb{Q}$-group,
- $G = \mathbb{G}(\mathbb{R})$,
- $X = \Gamma \setminus G$ where $\Gamma < \mathbb{G}(\mathbb{Q})$ congruence lattice, and
- $H < G$ connected semisimple subgroup without compact factors.

In this setup, the H-representations $L^2(xH)$ for $x \in X$ have a *uniform* spectral gap (Clozel's property (τ)).

Theorem (Einsiedler, Margulis, Venkatesh '09)

Assume that H has finite centralizer in G. There exist $\kappa > 0$, $d > 0$ depending on G, H and $V_0 > 0$ depending on G, H, Γ with the following property.

Suppose that xH is a closed H-orbit. For any $V \geq V_0$ there exists an intermediate group $H < S < G$ so that xS has finite volume $\mathrm{vol}(xS) \leq V$ and

$$\left| \int_{xH} f - \int_{xS} f \right| \leq V^{-\kappa} \mathcal{S}_d(f)$$

where $\mathcal{S}_d$ is an L^2-Sobolev norm of degree d.

The centralizer assumption

Generalizations: (at least) two directions:

- $\rightarrow$ subgroups H with infinite centralizer.
- $\rightarrow$ adelic analogues.

Together, the realm of applications – particularly in number theory – is very rich. We discuss the centralizer assumption here.

Existing generalizations in special cases:

- Aka, Einsiedler, Li and Mohammadi '20:

$$G = \mathrm{SL}_n(\mathbb{R}), \quad H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{SL}_k(\mathbb{R}), B \in \mathrm{SL}_{n-k}(\mathbb{R}) \right\}$$

  for $n - k \geq 3$.
- Einsielder and Wirth '20: $G = \mathrm{SO}(n,1)$, $H = \mathrm{SO}(2,1)$.

Here: We remove the centralizer assumption under a Diophantine condition.

Definition

We say that $\mathbb{M} < \mathbb{G}$ is of class-$\mathcal{H}$ if its radical is unipotent.

$$\mathcal{H} = \{\mathbb{M} < \mathbb{G} : \mathbb{M} \text{ has class-}\mathcal{H}\}$$

Theorem (W.)

There exist $\kappa > 0$, $\delta_1, \delta_2 > 0$, *and* $d > 0$ *depending on* G *and* H *with the following property.*
Let $x = \Gamma g \in X$ *such that* xH *is closed. Suppose that for all* $\mathbb{M} \in \mathcal{H}$ *with* $gHg^{-1} \subset \mathbb{M}(\mathbb{R})$ *and* $\mathbb{M} \neq \mathbb{G}$ *we have*

$$\mathrm{vol}(\Gamma \mathbb{M}(\mathbb{R}) g) > \mathrm{vol}(xH)^{\kappa}.$$

Then for all $f \in C_c^\infty(X)$

$$\left| \int f \, \mathrm{d}\mu_{xH} - \int f \, \mathrm{d}\mu_{xG^\circ} \right| \ll \mathrm{minht}(xH)^{\delta_2} \, \mathrm{vol}(xH)^{-\delta_1} \mathcal{S}_d(f)$$

where $\mathcal{S}_d$ *is an* L^2*-Sobolev norm.*

Thanks!