Rotated Odometers

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Rotated odometers

Rotated odometers are a class of infinite interval exchange transformations (IET).

Motivations for study:

- Can be considered as perturbation of the von Neumann-Kakutani map (standard dyadic odometer).

Methods: Bratteli-Vershik (adic) systems.

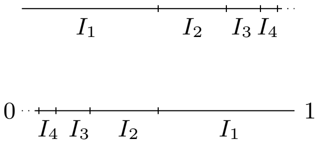
Results: Some results towards the classification up to an isomorphism (measure-theoretical and continuous factors).

Based on
H. Bruin and O. Lukina, *Rotated odometers*, arXiv: 2101.00868.

Let $I = [0, 1)$. The von Neumann-Kakutani map $A : I \to I$

$$A(x) = x - (1 - 3 \cdot 2^{1-n}) \qquad \text{if } x \in [1 - 2^{1-n}, 1 - 2^{-n}),\ n \geq 1,$$

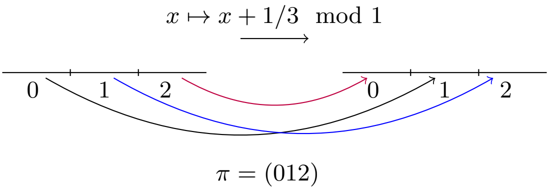
is an infinite IET.



The map A has a countable number of discontinuities at the points

$$\{1 - 2^{-n} \mid n \geq 1\}.$$

Let $q \geq 2$, and consider a rotation: $x \mapsto x + p/q \mod 1$.



This induces a permutation π and a finite IET $R_\pi : I \to I$.

A **rotated odometer** is an infinite IET

$$F_\pi = A \circ R_\pi : I \to I$$

A more general setting

Let $q \geq 2$, let π be **ANY** permutation of q symbols.

Let $R_\pi : I \to I$ be the corresponding finite IET (on intervals of equal length), and let

$$F_\pi = A \circ R_\pi : I \to I$$

be a rotated odometer.

Problem

What properties of the von Neumann-Kakutani map (I, A) are inherited by the rotated odometer (I, F_π)?

Method: choose a sequence of Poincaré sections (L_k, F_k), code the orbits of points in L_k under F_{k-1} by a partition of L_{k-1} into q sets.

Results:

- for each $k \geq 1$, F_k is a composition of a permutation R_k of q intervals, and a scaled von Neumann-Kakutani map.

- $I = I_{per} \cup I_{np}$, where I_{per} is a (possibly infinite) union of intervals of periodic points, I_{np} consists of non-periodic points.

- $0 \in I_{np}$, and the orbit of every non-periodic point accumulates at 0.

Lemma

The aperiodic subsystem (I_{np}, F_π) has a unique minimal set.

We now concentrate on the study of (I_{np}, F_π).

The coding of orbits produces a sequence $\{\chi_k\}_{k\geq 1}$ of substitutions on q symbols, such that:

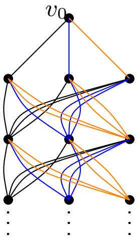
- every word $\chi_k(i)$ starts with 0,

- for a given k every word $\chi_k(i)$, $0 \leq i \leq q-1$, ends with the same letter.

Lemma: $\{\chi_k\}_{k\geq 1}$ is eventually periodic.

Associated to $\{\chi_k\}_{k\geq 1}$, there is an eventually stationary Bratteli diagram (V, E) with unique minimal and maximal paths.

Consequence: (V, E) admits a Vershik system (X, τ).



Theorem:
Possibly by passing to a subdiagram of (V, E), there is a measurable isomorphism

$$\psi : (I_{np}, F_\pi) \to (X, \tau)$$

of the aperiodic subsystem (I_{np}, F_π) of the rotated odometer to a Bratteli-Vershik system (X, τ).

Remark:
There is also a subdiagram $(\widehat{V}, \widehat{E})$ with Bratteli-Vershik system $(\widehat{X}, \widehat{\tau})$ which is measurably isomorphic under ψ to the minimal subsystem (I_{min}, F_π) of (I_{np}, F_π).

Instead of studying dynamical systems with discontinuities on an interval, we now can study dynamical systems given by a homeomorphism of a Cantor set.

We study the ergodic properties and the spectrum of the Koopman operator for these systems, using **Durand 2000**, **Ferenczi, Mauduit, Nogueira 1996**, **Bezuglyi, Kwiatkowski, Medynets, Solomyak 2010**, ...

Results (Bruin and Lukina 2021):

- The minimal subsystem $(\widehat{X}, \widehat{\tau})$ is uniquely ergodic.

- The aperiodic subsystem (X, τ) has at most q invariant ergodic measures.

- The Lebesgue measure on I is ergodic if and only if (I, F_π) has no periodic points.

- There exist infinitely many $q \geq 3$ and permutations π of q symbols, such that the minimal system $(\widehat{X}, \widehat{\tau})$ factors onto the dyadic odometer.

- There exist infinitely many $q \geq 3$ and permutations π of q symbols, such that the minimal system $(\widehat{X}, \widehat{\tau})$ does not factor onto the dyadic odometer, but is not weakly mixing.

- Similarly, there are examples where (X, τ) does or does not factor onto the dyadic odometer, but we did not find any weakly mixing examples.

Example: Let $q = 5$ and $\pi = (01234)$, then $\{\chi_k\}_{k\geq 1}$ is constant.

The substitution matrices for the diagrams are

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 8 & 8 & 4 & 8 \end{pmatrix} \text{ with } B_{min} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

B_{min} has eigenvalues 0 and 2, B has eigenvalues 8, 2 and 0 of multiplicity 3, so the Koopman operator has eigenvalues $e^{2\pi i s/2^r}$, $r \geq 1$.

Proposition:

- The minimal system $(\widehat{X}, \widehat{\tau}, \mu_1)$ is conjugate to the dyadic odometer.

- The dyadic odometer is the maximal equicontinuous factor for the full aperiodic subsystem (X, τ, μ_2).

Example: Let $q = 5$ and let $\pi = (02431)$, then $\{\chi_k\}_{k\geq 1}$ is constant.

$$B = \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 5 & 3 & 8 & 5 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \text{ with } B_{min} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

B has eigenvalues 8, $2 \pm \sqrt{5}$, -1 of multiplicity 2, B_{min} has eigenvalues $2 \pm \sqrt{5}$, -1 of multiplicity 2.

The Koopman operator of the minimal system $(\widehat{X}, \widehat{\tau}, \mu_1)$ has eigenvalues $e^{2\pi s(a+b\sqrt{5})}$, $a, b \in \mathbb{Q}$, $s \in \mathbb{Z}$.

The Koopman operator of the aperiodic system (X, τ, μ_2) has eigenvalues $e^{2\pi i/2}$, $e^{2\pi i/4}$.

We have found examples of rotated odometers, where the minimal system $(\widehat{X}, \widehat{\tau})$ and the aperiodic subsystem (X, τ) do or do not factor onto the dyadic odometer.

But we can only determine that on a case-by-case basis.

Problem

Let $F_\pi : I \to I$ be a rotated odometer. Find necessary and sufficient conditions under which $(\widehat{X}, \widehat{\tau})$ (or (X, τ)) has the dyadic odometer as a factor.

Problem

Are there any rotated odometers for which Lebesgue or the measure on I_{min} are weakly mixing (i.e. 1 is the only eigenvalue of the Koopman operator)?