Continued fraction normality for the Minkowski Question Mark Function

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Section 1

Background

Normal numbers

Let $b \in \mathbb{N} + 1$ be a "radix" *base*.

Let $x = .x_1x_2x_3 \cdots \in [0, 1)$, $x_i \in \{0, 1, \ldots, b-1\}$, be the base b expansion of x:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{b^k}.$$

Definition

Call x a *base-b normal number* if for any $w \in \mathcal{D}_b^*$,

$$\lim_{n \to \infty} \frac{N_n(x, w)}{n} = \frac{1}{b^{|w|}},$$

where $N_n(x, w)$ is the number of times w occurs in the first n digits of x.

History

1. Émil Borel, 1909: Lebesgue a.e., $x \in [0,1)$ is normal in every base.
2. Not proved not to be normal: $x = \pi, \sqrt{2}, e, \ldots$.
3. Champernown, 1933: explicit base $b = 2$ normal number

$$c_{10} = 0.123456789\ 101112131415161718\ldots.$$

4. Base $b = 2$ version:

$$c_2 = 0.1\ 1011\ 100101110111\ 1000100110101011\ldots$$

Alternative $b = 2$ version:

$$c_2' = 0.0\ 01\ 00011011\ 000001010011100101110111\ 0000\ldots$$

Radix transformation

$T_b : [0, 1) \to [0, 1)$ defined

$$T_b x := bx \bmod 1 \text{ or equivalently } T_b(.x_1 x_2 x_3 \dots) = .x_2 x_3 x_4 \dots$$

ergodic Lebesgue measure λ preserving, full b-shift, $h(T_b) = \log b$.

By the ergodic theorem, a.e. x satisfies

$$\lim_{n\to\infty} \frac{N_n(x, w)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\Delta(w)}(T_b^k x) = \lambda(\Delta(w)) = \frac{1}{b^{|w|}},$$

where $\Delta(w) = \{x : .x_1 x_2 \dots x_{|w|} = .w_1 w_2 \dots w_{|w|}\}$.

This proves almost every x is normal base b (and in every base).

Section 2

Continued fractions

Expansions

Any $x \in (0,1)\backslash\mathbb{Q}$ has a unique infinite *continued fraction expansion*

$$x = [a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

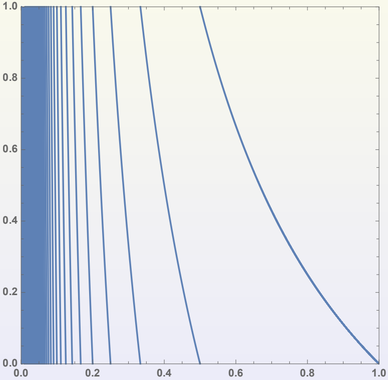
$a_k \in \mathbb{N}$.

The fundamental interval is

$$\Delta([w_1, w_2, \dots, w_n]) = \{x : [a_1, a_2, \dots, a_m] = [w_1, w_2, \dots, w_n]\}.$$

Gauss transformation

Let $G : [0,1] \to [0,1]$ be defined $Gx := 1/x \mod 1$.



The *Gauss measure*, $d\gamma = \rho(x) d\lambda$ where $\rho(x) = \frac{1}{\ln 2} \frac{1}{x+1}$ is ergodic G-invariant. Also $h_\gamma(G) = \frac{\pi^2}{6 \ln 2} \sim 2.37314$.

Continued fraction normality

Definition

A number $x \in [0,1]$ is *Gauss continued fraction normal* if for each $w \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \frac{N_n(x,w)}{n} = \gamma(\Delta(w)).$$

By the ergodic theorem:

Theorem

Almost every $x \in (0,1)$ is Gauss continued fraction normal.

$$\lim_{n\to\infty} \frac{N_n(x,w)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\Delta(w)}(G^k x) = \gamma(\Delta(w)).$$

Adler-Keane-Smorodinski

Consider *this* enumeration of the (unreduced) rationals

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \quad \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \quad \cdots, \frac{1}{n}, \ldots, \frac{n-1}{n}, \quad \frac{1}{n+1}, \cdots$$

Concatenate their (finite) continued fractions:

$$\begin{aligned} s &= [2] \quad [3][1,2] \quad [4][2][1,3] \quad [5][2,2][1,1,2][1,4] \cdots \\ &= [2,3,1,2,4,2,1,3,5,2,2,1,1,2,1,4,\cdots] \approx 0.44034. \end{aligned}$$

Theorem (Adler, Keane & Smorodinski, 1981)

The number s is Gauss continued fraction normal.

Section 3

The Minkowski "question mark" function

Definition

The Minkowski (1904) *"question mark" function* is a strictly increasing bijection

$$Q : \mathbb{Q} \cap [0,1] \to \mathbb{Z}[1/2] \cap [0,1].$$

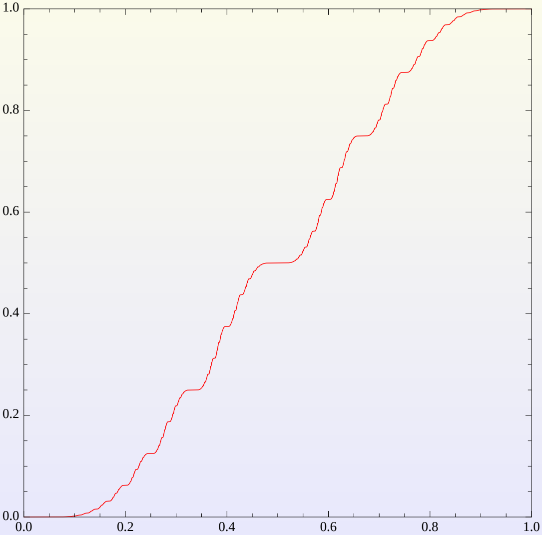
Defined inductively: $Q(0) = 0$, $Q(1) = 1$ and

$$Q\left(\frac{p}{q} \oplus \frac{r}{s}\right) := Q\left(\frac{p+r}{q+s}\right) = \frac{Q(p/q) + Q(r/s)}{2}$$

using *Farey sums: Farey (1816) called them "vulgar fractions"*.

Extends to continuous, strictly increasing (bijection) $Q : [0,1] \to [0,1]$.
Denjoy, 1938: $Q'(x) = 0$ a.e..

Graph



Closed form

$$Q(x) := 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+a_2+\cdots+a_i}},$$

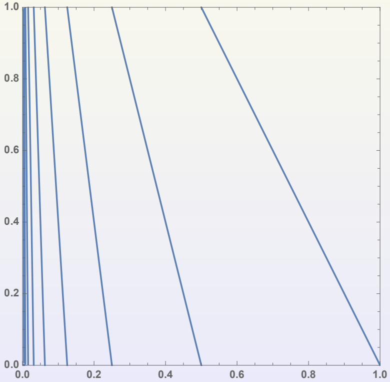
where $x = [a_1, a_2, a_3, \ldots]$.

The *Minkowski question mark measure*, defined $q((a,b)) := Q(b) - Q(a)$, is positive on open sets, and satisfies $q \perp \lambda$.

Theorem

The Gauss transformation G preserves q and is ergodic with entropy is $h_\gamma(G) = 2\ln 2 \sim 1.38629$.

Define the *generalized Lüroth transformation* $L(x) = 2 - 2^n x$ for $x \in (1/2^n, 1/2^{n-1}]$.



L is ergodic, Legesgue measure preserving, and has $h_\lambda(L) = 2\log 2$.

Proposition (*)

$Q \circ G = L \circ Q$ and $Q^* q = \lambda$.

For the latter,

$$(Q^* q)([0, x)) = q([0, Q^{-1}(x))) = QQ^{-1}(x) = x = \lambda([0, x)).$$

A related fact is

Lemma

For a fundamental interval

$$q(\Delta(w)) = 2^{-(a_1 + a_2 + \cdots + a_\ell)}.$$

Minkowski-? normality

Definition

A number $x \in [0,1]$ is *Minkowski question mark normal* if for each $w \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{N_n(x,w)}{n} = q(\Delta(w)).$$

Corollary

Almost every (for q) $x \in [0,1]$ is Minkowski question mark normal.

Section 4

A Minkowski normal number

Kepler tree

Start with $\frac{1}{2}$. Make infinite binary tree of all (reduced) $\frac{p}{q} \in (0,1) \cap \mathbb{Q}$

$$p/q \quad \to \quad p/(p+q) \qquad q/(p+q)$$

Johannes Kepler: *Harmonices Mundi*, (1621).

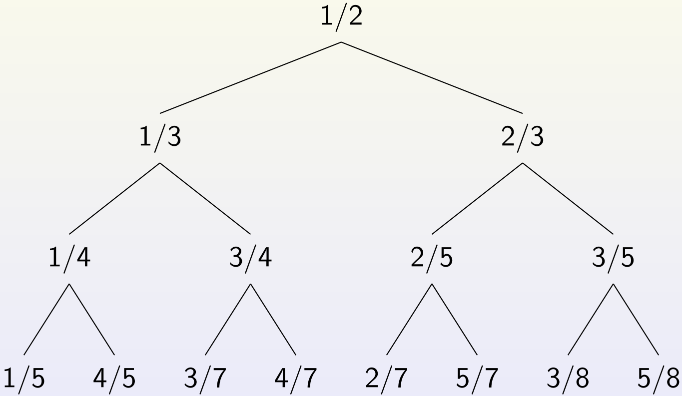
Theorem

Every $p/q \in (0,1) \cap \mathbb{Q}$, in lowest terms, occurs exactly once.

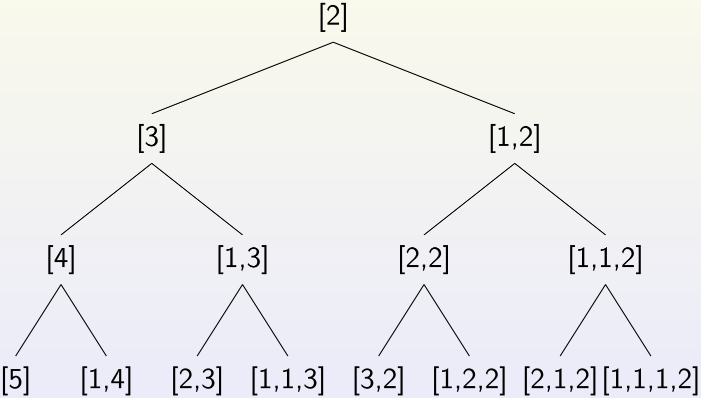
As continued fractions

$$[a_1, a_2, \cdots, a_\ell] \quad \to \quad [(a_1+1), a_2, \cdots, a_\ell] \qquad [1, a_1, a_2, \cdots, a_\ell].$$

First four levels (rationals)



First four levels (continued fractions)



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The ?-normal number

Enumerate the rationals along rows of the Kepler tree

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \quad \frac{1}{5}, \frac{4}{5}, \frac{3}{7}, \frac{4}{7}, \frac{2}{7}, \frac{5}{7}, \frac{3}{8}, \frac{5}{8}, \dots .$$

Concatenate the (finite) continued fractions of these rationals:

$$\begin{aligned} k &= [2]\ [3][1,2]\ [4][1,3][2,2][1,1,2]\ [5][1,4][2,3][1,1,3][3,2][1,2,2]\cdots \\ &= [2,3,1,2,4,1,3,2,2,1,1,2,5,1,4,2,3,1,1,3,3,2,1,\cdots] \approx 0.44031. \end{aligned}$$

Theorem (Dajani, de Lepper, R, 2019)

The number k is Minkowski question mark continued fraction normal.

Set up for proof

Four types of blocks $w = [w_1, w_2, \ldots, w_\ell] \in \mathbb{N}^*$ in k:

1. beginning (of rational in the Kepler tree),
2. middle,
3. end, or
4. across.

Type 4 is *"divided occurrence"*: these have density zero.

Now count other types.

Section 5

Down the Kepler tree

Kepler tree on continued fractions

Corollary (of Kepler's tree)

Every block $[w_1, w_2, \ldots, w_\ell] \in \mathbb{N}^*$ ($a_\ell \geq 2$), equivalently $p/q \in (0,1) \cap \mathbb{Q}$, occurs once in the Kepler tree.

Code paths down Kepler tree: 0="keep left", 1="keep right".

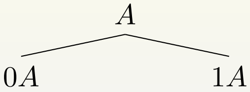
Proposition

Suppose $p/q = [w_1, w_2, \ldots, w_\ell] \in (0,1) \cap \mathbb{Q}$. Then the path to p/q is $0^{w_\ell - 2} 1 \cdots 1 0^{w_2 - 1} 1 0^{w_1 - 1}$.

Since $w_j \geq 1$ and $w_\ell \geq 2$, $w_j - 1, w_\ell - 2 \in \mathbb{N} \cup \{0\}$.

The binary tree

The tree on binary paths $A \in \{0,1\}^*$ branches like this:



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We start with $A = \varepsilon$ (the empty word) at the root.

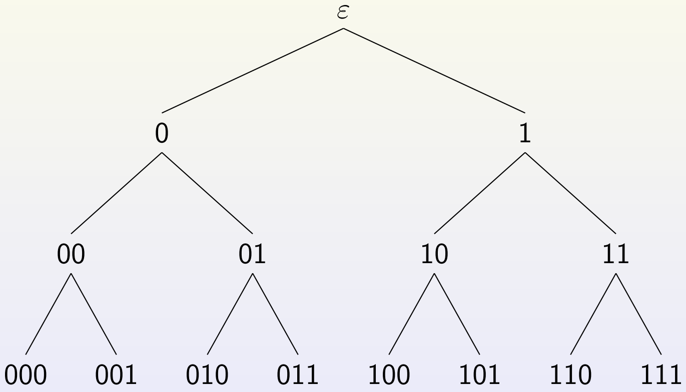
The alternative binary Champernowne number

$$c_2' = 0.0\ 01\ 00011011\ 00000101001\ldots,$$

is obtained by concatenating these binary blocks, level-by level, in the same order as the continued fraction blocks in κ (and replacing ϵ by 0).

The length of this concatenation at level L is given by $B(L) = L2^L$

First four levels (binary)



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Blocks

Proposition

There is a 1:1 correspondence between undivided occurrences of $[w_1, w_2, \ldots, w_\ell]$ in k and undivided occurrences of $w = 100^{w_\ell - 2} \cdots 10^{w_2 - 1} 10^{w_1 - 1} 1$ in c_2'.

The frequency of w in b_2' is $2^{-|w|} = 2^{-(w_1 + w_2 + \cdots + w_\ell)}$.

So the frequency of $[w_1, w_2, \ldots, w_\ell]$ in k is

$$(1/2) 2^{-(w_1 + w_2 + \cdots + w_\ell)} = (1/2) q(\Delta([w_1, w_2, \ldots, w_\ell])).$$

It follows that k is Minkowski question mark normal. □

Levels

The number of moves down the tree in the path $0^{a_\ell - 2} \cdots 10^{a_2-1}10^{a_1-1}$ (to $[a_1, a_2, \ldots, a_\ell]$) is $L = a_1 + a_2 + \cdots + a_\ell - 2$.

Corollary

The blocks $[a_1, a_2, \ldots, a_\ell]$ at level $L \geq 0$ are **all** the blocks that satisfy $L = a_1 + a_2 + \cdots + a_\ell - 2$. These correspond to the binary codes $b \in \{0,1\}^L$ in lexicographic order.

To make κ we first concatenate the blocks at each level. The lengths of these level-L concatenated blocks satisfies the recurrence relation

$$A(L+1) = 2A(L) + 2^L \text{ with } A(0) = 1.$$

The solution is $A(L) = (L+2)2^{L-1}$.

Denominator

On the other hand, The length of level $L > 0$ of the concatenated blocks in the binary tree is $B(L) = L2^L$.

Lemma

One has

$$\lim_{L\to\infty} \frac{B(L)}{A(L)} = \lim_{L\to\infty} \frac{L2^L}{(L+1)2^{L-1}} = 2 \lim_{L\to\infty} \frac{L}{L+1} = 2.$$

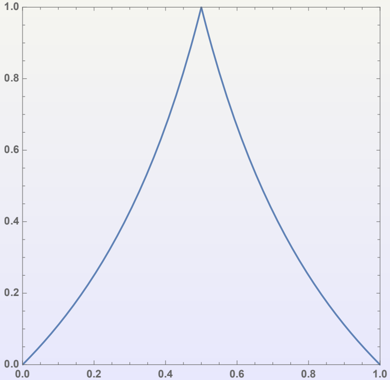
Section 6

The Farey transformation

Definition

The *Farey transformation* $F : [0,1] \to [0,1]$ is defined

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \le \frac{1}{2}, \\ \frac{1-x}{x} & \text{if } x \ge \frac{1}{2}. \end{cases}$$



- Satisfies $F'(0) = 1$ (non-hyperbolic). No absolutely continuous *finite* invariant measure.
- Absolutely continuous *infinite* invariant measure, with density $\delta(x) = 1/x$.
- $G(x) = F^{a_1 - 1}(x)$ (called the "Jump" by Schweiger). F is "slow continued fractions".

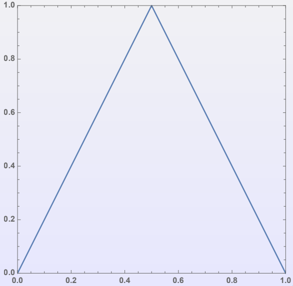
Just like Proposition (*), we have

Proposition (**)

$Q \circ F = T \circ Q$ and $Q^* q = \lambda$.

Here, T is the λ-preserving full tent map:

$$T(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2}, \\ 2 - 2x & \text{if } x \geq \frac{1}{2}. \end{cases}$$



Binary

The tent map T is isomorphic to the one-sided 2-shift. So the binary Champernowne number c_2' (or c_2) is normal for T.

Corollary

The number k is Minkowski question mark normal for the Farey map F.

Corollary

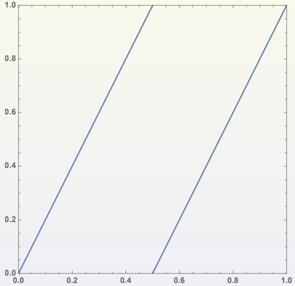
The Minkowski question mark measure is the (unique finite) measure of maximal entropy for the Farey map F. In particular, $h_q(F) = \log 2$.

Section 7

Binary again

Fair coin

The *angle doubling map* $D : [0,1] \to [0,1]$ is defined $D(x) = 2x \mod 1$.



It is isomorphic to the one-sided 2-shift. So the binary Champernowne number, say c_2', is normal for D.

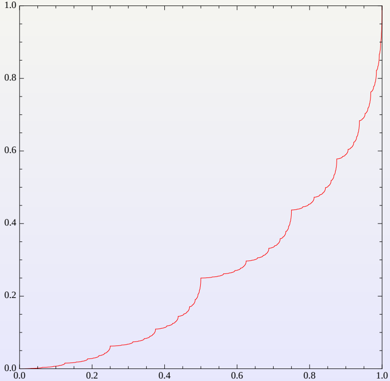
- Here λ corresponds to the "fair coin" Bernoulli measure $\beta_{1/2} = \prod_{n \in \mathbb{N}} \delta_{1/2,1/2}$.

Unfair coin

What about an unfair coin, say $\beta_{1/3} = \prod_{n \in \mathbb{N}} \delta_{1/3,2/3}$?

Let $S(x)$ be the distribution function for this Bernoulli measure

The function $S(x)$ (studied by Raphael Salem, 1943) is continuous, strictly-increasing, and $S'(x) = 0$ a.e. λ



- Take a base-3 Champernowne number and locate all the 2s:

$$c_3 = .0\ 1\ 2\ 00\ 01\ 02\ 10\ 11\ 12\ 20\ 21\ 22\ 000\ 001\ 002\ldots.$$

- Change 2s to 1s

$$u_{1/3} = .0\ 1\ 1\ 00\ 01\ 01\ 10\ 11\ 11\ 10\ 11\ 11\ 000\ 001\ 001\ldots.$$

- Then $u_{1/3}$ is normal for the unfair coin.
- Same idea for u_α for any $\alpha \in \mathbb{Q}$.

Question: Make u_α for $\alpha \notin \mathbb{Q}$?

Reference

Karma Dajani, Mathijs de Lepper and R, *Introducing Minkowski Normality*, Journal of Number Theory, **211**, (2020), pp 445-476.

Thank You!