Classification of topological systems which split into uniquely ergodic subsystems

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based on a joint work with

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Nevertheless, we found it interesting to classify systems which satisfy this condition (we called them *pointwise ergodic*).

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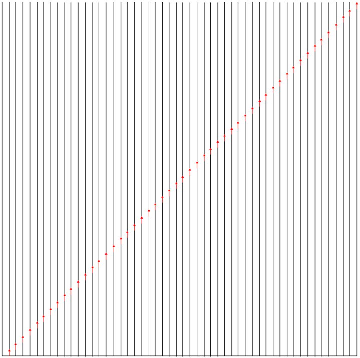

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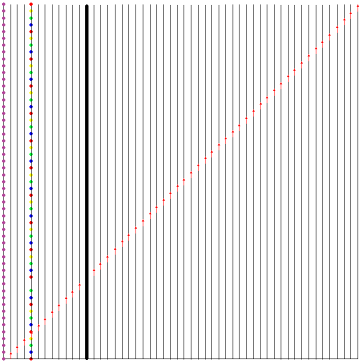

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Can strict pointwise genericity be reasonably weakened or strengthened within the class of pointwise generic systems?

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(VI) *Continuously strictly pointwise ergodic* if it is continuously pointwise ergodic and semi-simple.

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(S) *Continuously supported*, if the set-valued function $\mathrm{supp} : \mathcal{M}^{\mathrm{e}}_\sigma(X) \to 2^X$, where $\mathrm{supp}(\mu)$ denotes the topological support of a measure, is continuous in the Hausdorff metric.

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(I)-p.erg., (II)- semis., (III)-part., (IV)-str.part., (V)-cont.p.erg., (VI)-cont.str.p.erg., (R)-reg.p.erg., (B)-bauer., (S)-cont.supp.

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(sU) *Strictly uniform* if it is both uniform and semi-simple.

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What we find interesting, is that uniformity (or strict uniformity) which has an ergodic-theoretic flavor, can be expressed in terms of topological properties of the partition of the system into uniquely (strictly) ergodic components.

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Surprisingly, it is not the case. As we will show, neither (III)$\wedge$(B)$\wedge$(S) implies (V), nor (V) implies (S).

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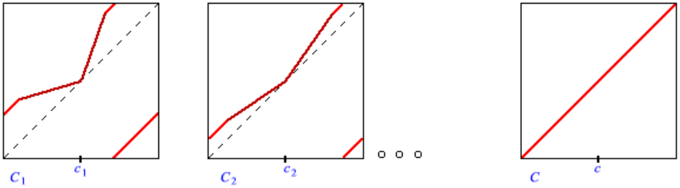

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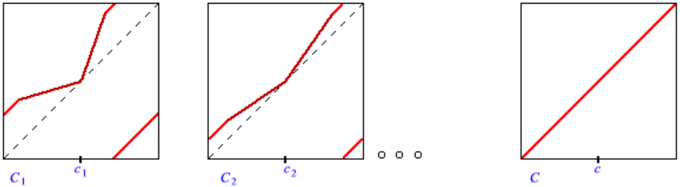

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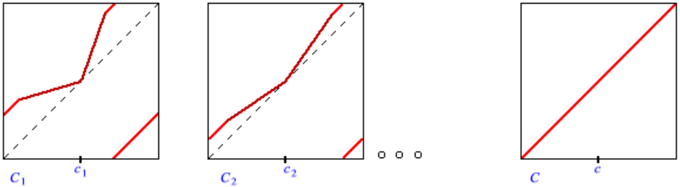

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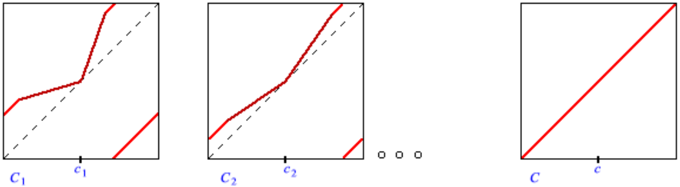

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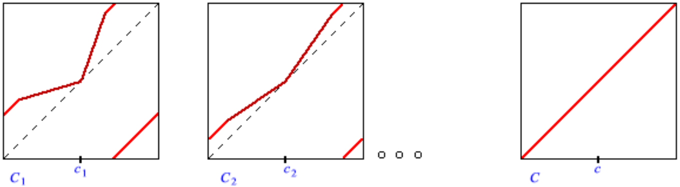

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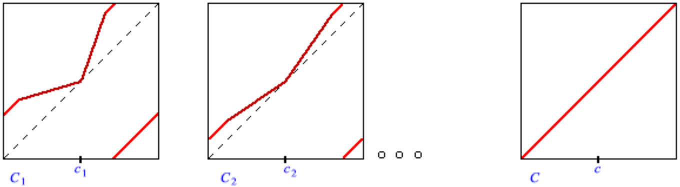

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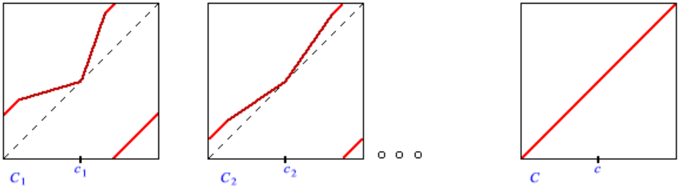

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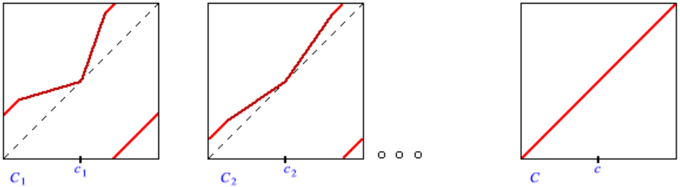

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The system is partitionable (III).
The atoms of $\mathcal{P}$ are the circles C_k and single points on C, so $\mathcal{P}$ is not upper semicontinuous, hence system is **not** continuously pointwise ergodic (¬V).

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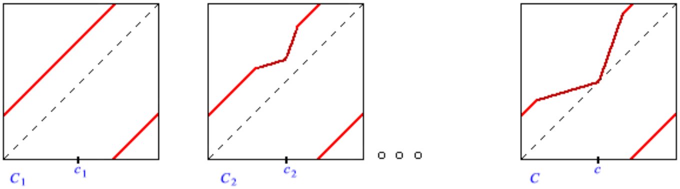

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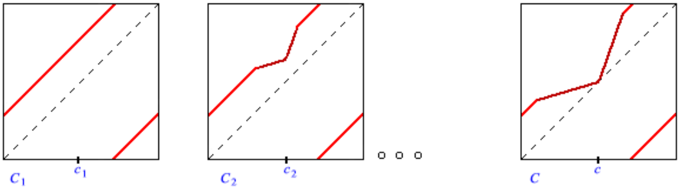

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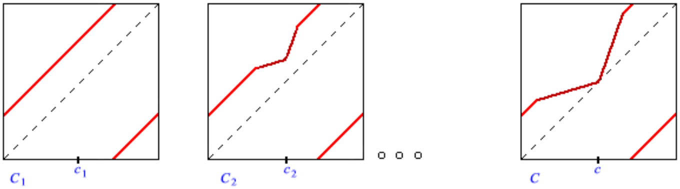

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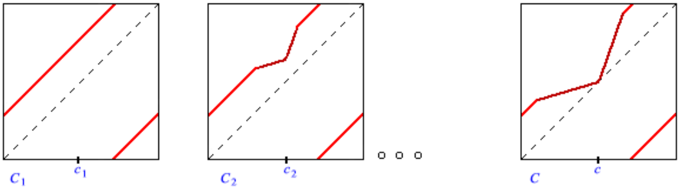

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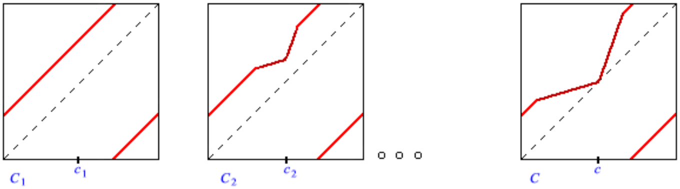

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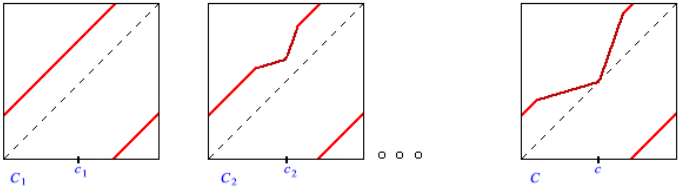

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This implies that $\mathcal{M}_\sigma^{\mathrm{e}}(X)$ is compact (B).
However, the supports $\mathrm{supp}(\mu_k)$ are circles, while $\mathrm{supp}(\delta_c)$ is one point, hence the function $\mathrm{supp} : \mathcal{M}_\sigma^{\mathrm{e}}(X) \to 2^X$ is **discontinuous** $(\neg \mathrm{S})$.

That's all, thank you!