Descent moduli and metric determination of functions

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Abstract. The fundamental theorem of calculus asserts that, for every two differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ one has that

$$\nabla f(x) = \nabla g(x), \forall x \in \mathbb{R}^n \implies f = g + c,$$

for some constant $c \in \mathbb{R}$. This is what is called a *determination result.* Recently, in [5], a very impressive determination result was obtained in the Hilbert space setting: for every two convex functions $f, g : \mathcal{H} \to \mathbb{R}$ over a Hilbert space $\mathcal{H}$, of class $\mathcal{C}^2$ and bounded from below, we have that

$$\|\nabla f(x)\| = \|\nabla g(x)\|, \forall x \in \mathbb{R}^n \implies f = g + c,$$

where $c = \inf_{\mathcal{H}} f - \inf_{\mathcal{H}} g$. This pioneer result shows that, in the case of $\mathcal{C}^2$ convex functions that are bounded from below, the slope (norm of the gradient, scalar information) is enough information in order to determine a function, strongly improving the fundamental theorem of calculus which requires the whole gradient (vectorial information).

In this talk, we will revise the latest developments in the line of determination using slopes: we first visit extensions to continuous convex functions [4, 6]. Then, we will explore nonconvex continuous functions in compact metric spaces [3], and abstract notions of slopes known as *Descent Moduli* [2]. Finally, we discuss how a Descent Modulus can be used to determine continuous functions in complete metric spaces, freeing from the previous convexity/compactness assumptions [1].

References

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