Metastability for small random perturbations of a PDE with blow-up

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The PDE and its small random perturbations

For $u \in C_D([0,1]) = \{v \in C([0,1]) : v(0) = v(1) = 0\}$, let U^u be the solution of the PDE

$$\begin{cases} \partial_t U^u &= \partial^2_{xx} U^u + g(U^u) & t > 0\,,\ 0 < x < 1 \\ U^u(t,0) &= 0 & t > 0 \\ U^u(t,1) &= 0 & t > 0 \\ U^u(0,x) &= u(x). & 0 < x < 1 \end{cases}$$

where $g(u) = u|u|^{p-1}$ for fixed $p > 1$.

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where $g(u) = u|u|^{p-1}$ for fixed $p > 1$. For each $\varepsilon > 0$ we consider the random perturbation $U^{u,\varepsilon}$ of U^u given by

$$\begin{cases} \partial_t U^{u,\varepsilon} & = \partial^2_{xx} U^{u,\varepsilon} + g(U^{u,\varepsilon}) + \varepsilon \dot{W} & \\ U^{u,\varepsilon}(t,0) & = 0 & t > 0 \\ U^{u,\varepsilon}(t,1) & = 0 & t > 0 \\ U^{u,\varepsilon}(0,x) & = u(x). & 0 < x < 1 \end{cases}$$

where $\dot{W}$ is space-time white noise.

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- Background in metastability for small random perturbations of differential equations:
  - Escape from domain with unique attractor : Day ('83)
  - Double-well potential models
    - ODE: Galves,Olivieri,Vares ('87)
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◇ Little is known about blow-up in stochastic systems.

The gradient formulation

The PDE can be reformulated as

$$\partial_t U^u = -\nabla S(U^u)$$

where $S : C_D([0,1]) \to (-\infty, +\infty]$ is the potential given by

$$S(v) = \begin{cases} \int_0^1 \left[\frac{1}{2} \left(\frac{dv}{dx} \right)^2 - \frac{|v|^{p+1}}{p+1} \right] & \text{if } v \in H_0^1((0,1)) \\ +\infty & \text{otherwise.} \end{cases}$$

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  - The remaining equilibriums are all saddle points.
- $z^{(1)} := z$ is the unique nonnegative unstable equilibrium.
- $z^{(-n)} = -z^{(n)}$ for every $n \in \mathbb{N}$ and

$$S(\mathbf{0}) < S(\pm z^{(1)}) < \cdots < S(\pm z^{(n)}) \nearrow +\infty$$

$$\|\mathbf{0}\|_\infty < \| \pm z^{(1)}\|_\infty < \cdots < \| \pm z^{(n)}\|_\infty \nearrow +\infty$$

The unstable equilibria

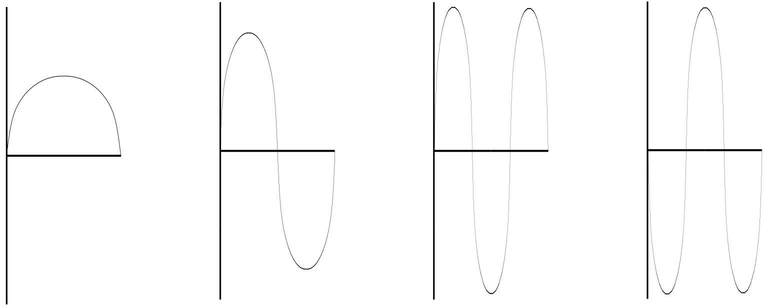

Figure 1: Unstable equilibria (left to right) : $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ and $z^{(-3)}$.

Behavior of U^u for different initial data $u \in C_D([0,1])$

Given $u \in C_D([0,1])$, the solution U^u can behave in two ways:

(I) U^u blows up in finite time: there exists $\tau^u < +\infty$ such that

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- The domain of explosion
  $\mathcal{D}_e := \{u : U^u \text{ blows up in finite time}\}$
- The basin of attraction
  $\mathcal{D}_{\mathbf{0}} := \{u : U^u \text{ is globally defined and } \lim_{t\to+\infty} U^u(t) = \mathbf{0}\}$.

Brownian sheet and the SPDE

For $\varepsilon > 0$ we consider $U^{u,\varepsilon}$ the solution of

$$\begin{cases} \partial_t U^{u,\varepsilon} & = (\partial^2_{xx} U^{u,\varepsilon} + U^{u,\varepsilon}|U^{u,\varepsilon}|^{p-1})dt + \varepsilon \dot{W} & \\ U^{u,\varepsilon}(t,0) & = 0 & t > 0 \\ U^{u,\varepsilon}(t,1) & = 0 & t > 0 \\ U^{u,\varepsilon}(0,x) & = u(x). & 0 < x < 1 \end{cases}$$

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- W is a Brownian sheet.

Heuristic scenario

Based on the previous considerations, for $u \in \mathcal{D}_{\mathbf{0}}$ one expects the following scenario:

- First, due to the action of the gradient $-\nabla S$, the process $U^{u,\varepsilon}$ is attracted towards $\mathbf{0}$.
- Once close to $\mathbf{0}$, the term $-\nabla S$ becomes negligible and the noise pushes $U^{u,\varepsilon}$ away from $\mathbf{0}$.
- Thermalization. Large number of attempts to escape from the "valley", followed by a strong attraction towards its bottom.
- Tunneling. After many frustrated attempts, $U^{u,\varepsilon}$ successfully escapes the valley and reaches $\mathcal{D}_e$. It will do so through $\pm z$, since the potential barrier is the lowest there.
- Final excursion. Once in $\mathcal{D}_e$, the noise is overpowered by the exploding source term and $U^{u,\varepsilon}$ blows up in finite time.

Results I

Theorem I. Let $\mathcal{D}_e^*$ be the set of initial data for which U^u blows up in finite time but only trough one side, i.e. U^u remains bounded either from below or above until the explosion time τ^u.

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Then, given $u \in \mathcal{D}_e^*$ and $\delta > 0$ there exists $C > 0$ such that

$$P(|\tau_\varepsilon^u - \tau^u| > \delta) \leq e^{-\frac{C}{\varepsilon^2}}$$

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Theorem II. If $1 < p < 5$ then, given $\rho > 0$ and $u \in \mathcal{D}_{\mathbf{0}}$,

$$\lim_{\varepsilon \to 0} P(\tau_\varepsilon^u(B(\pm z, \rho)) < \tau_\varepsilon^u) = 1.$$

where $\tau_\varepsilon^u(B(\pm z, \rho))$ is the hitting time of the set

$$B(\pm z, \rho) := B(z, \rho) \cup B(-z, \rho).$$

Results II

Theorem III. If $\Delta := 2(S(z) - S(\mathbf{0}))$ and $1 < p < 5$, there exists a sequence $(\beta_\varepsilon)_{\varepsilon>0} \subseteq \mathbb{R}_{>0}$ such that

i. $\lim_{\varepsilon\to 0} \varepsilon^2 \log \beta_\varepsilon = \Delta$, i.e. $\beta_\varepsilon \approx e^{\frac{\Delta}{\varepsilon^2}}$,

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$$\lim_{\varepsilon\to 0} P(\tau_\varepsilon^u > t\beta_\varepsilon) = e^{-t}$$

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Theorem IV. If $1 < p < 5$ then, given $\theta, \delta > 0$, for any $0 < \alpha < \Delta$ and $u \in \mathcal{D}_0$

$$\lim_{\varepsilon\to 0} P\left(\sup_{0\leq t\leq \tau_\varepsilon^u - 2R_\varepsilon} \left| \frac{1}{R_\varepsilon} \int_t^{t+R_\varepsilon} \mathbb{1}_{[-\theta,\theta]^c}(U^{u,\varepsilon}(s))ds \right| > \delta \right) = 0$$

where $R_\varepsilon = e^{\frac{\alpha}{\varepsilon^2}}$.

Proofs - some ideas I

Ideas for the proof of Theorem I.

- Consider the process $Z^{u,\varepsilon} := U^{u,\varepsilon} - V^{\mathbf{0},\varepsilon}$ where $V^{\mathbf{0},\varepsilon}$ is the solution of the SPDE

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- Idea. Study the explosion time of $Z^{u,\varepsilon}(\omega)$ for each $\omega \in \Omega$ with PDE techniques:

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- Idea. Study $\tau_\varepsilon^u(\partial G)$ by using large deviations estimates, (we may assume g Lipschitz to study $U^{u,\varepsilon}$ while inside G).

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