The contact process on a one-dimensional dynamical percolation environment

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Joint work with Daniel Remenik

The Contact Process

- It is a homogeneous Markov jump process $\{\eta_t\}_{t \geq 0}$ with state space

$$\{\eta : S \longrightarrow \{0, 1\}\}$$

S: vertices of a denumerable graph G

η: system configurations

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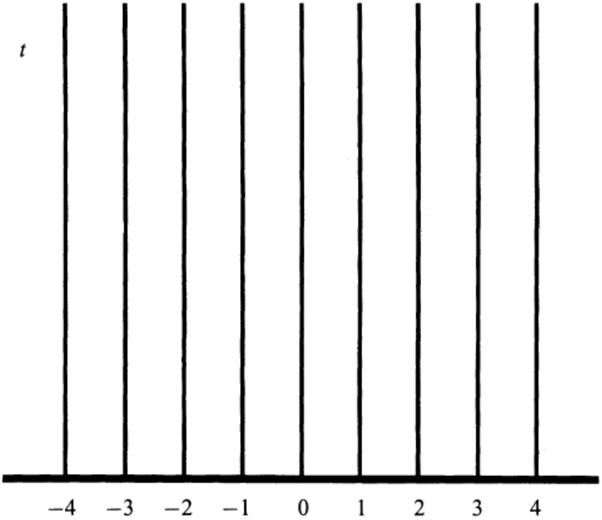
- For each $x \in S$, $\eta(x)$ jumps to $1 - \eta(x)$ with rate:

$$\begin{array}{lll} 1 \to 0 & \text{at rate} & 1 \\ 0 \to 1 & \text{at rate} & \lambda \sum_{y \sim x} \eta(y) \end{array}$$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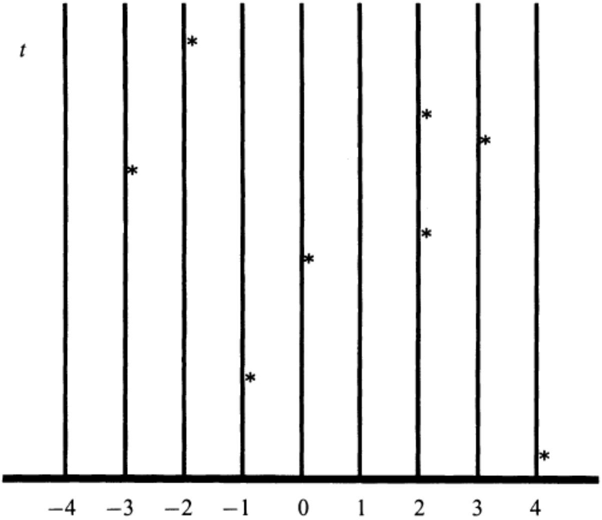
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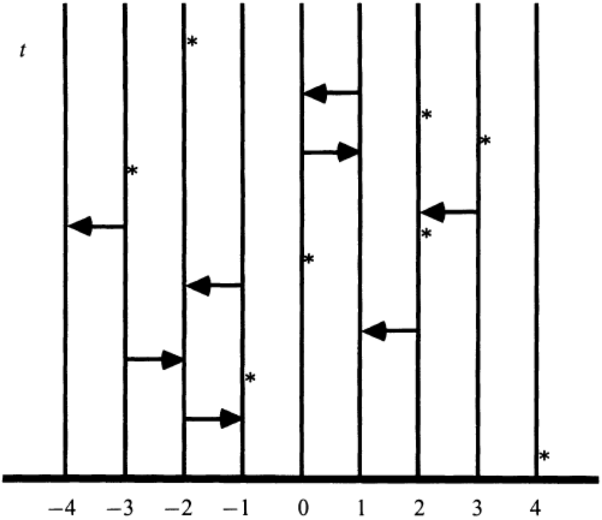

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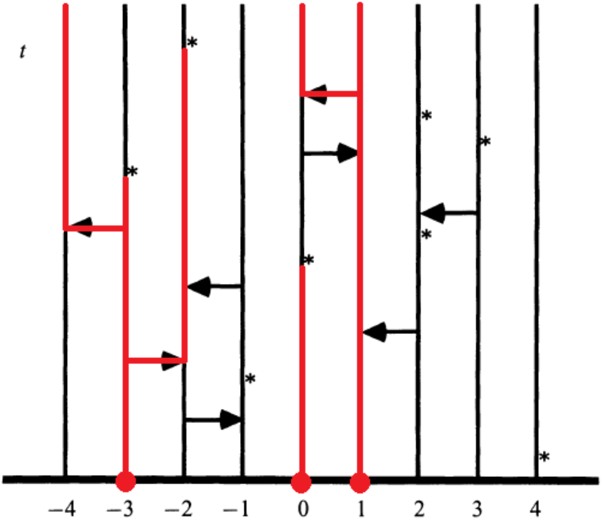
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The survival probability

$$\alpha_\eta^\lambda := \mathbb{P}_\eta^\lambda\big(\eta_t \neq \emptyset \; \forall t \geq 0\big)$$

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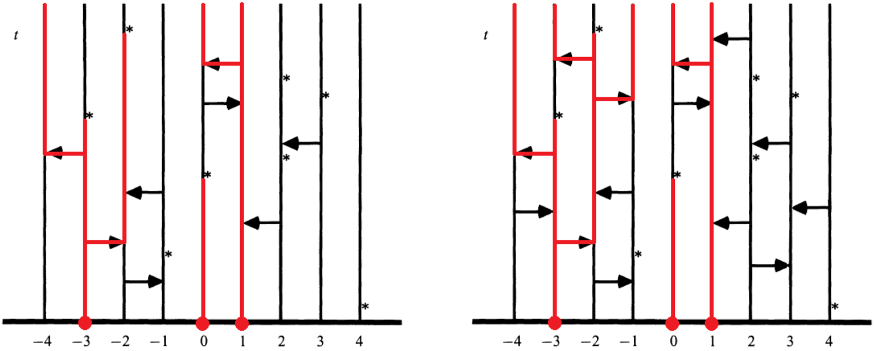

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- Phase transition: $\exists \lambda_0 \in [0, \infty]$ such that, $\forall |\eta| < \infty$;

$$\forall \lambda < \lambda_0, \quad \alpha_\eta = 0$$

$$\forall \lambda > \lambda_0, \quad \alpha_\eta > 0$$

Some properties:

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$$\lambda < \lambda_0 \iff \eta_t^S \xrightarrow{\mathcal{D}} 0$$

When $S = \mathbb{Z}^d$, Bezuidenhout and Grimmett (1990) developed a technique based on block construction and oriented percolation obtaining:

- The process dies out at $\lambda = \lambda_0$
- α_η^λ is a continuous function of λ.

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Pemantle (1992), adapting a technique used to study Branching Random Walk, proved the same results for the case $S = \mathbb{T}^d$.

Contact process on random graphs

Contact process on random graphs

- Chen and Yao (2008) studied the contact process on the half space $\mathbb{Z}^d \times \mathbb{Z}^+$ under supercritical bond percolation and showed that it behaves as in $\mathbb{Z}^d$.
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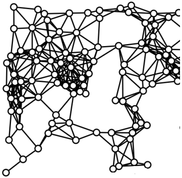
- Pemantle (2003) studied the process on Galton-Watson trees and proved that if the descendants distribution has a heavy enough tail then

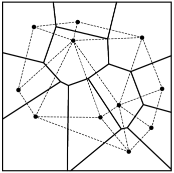
$$\lambda_0 = 0$$

Ménard y Singh (2015) developed the study of Cumulative Merging Percolation, obtaining methods to show that $\lambda_0 > 0$ for random graphs with unbounded degree such as:

- Supercritical Geometric graph



- Delaunay Triangulation



Dynamical bond percolation on $\mathbb{Z}$

Define $\{\xi_t(x,y)\}_{t\geq 0, x\sim y\in\mathbb{Z}}$ i.i.d continuous Markov process on $\{0,1\}$ with

$$\begin{array}{cccll} 0 & \to & 1 & \text{at rate} & vp \\ 1 & \to & 0 & \text{at rate} & v(1-p) \end{array}$$

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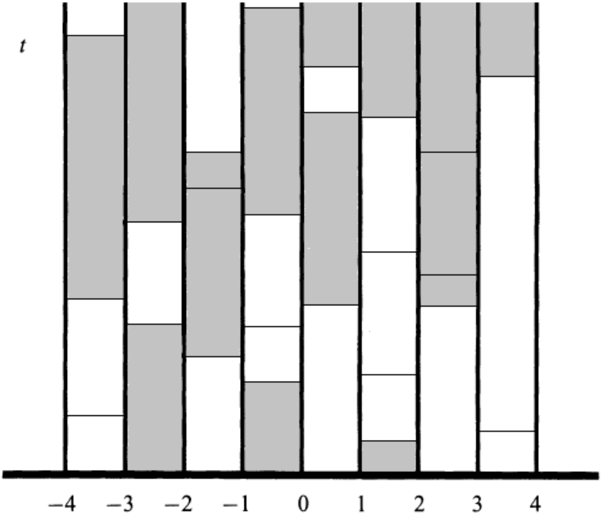
Define the Contact Process in Dynamical Percolation (CPDP) $\{\eta_t\}_{t\geq 0}$ as a contact process on $\mathbb{Z}$ with rates

$$\begin{array}{lll} 1\rightarrow 0 & \text{at rate} & 1 \\ 0\rightarrow 1 & \text{at rate} & \lambda \sum_{y\sim x} \eta(y)\xi_t(x,y) \end{array}$$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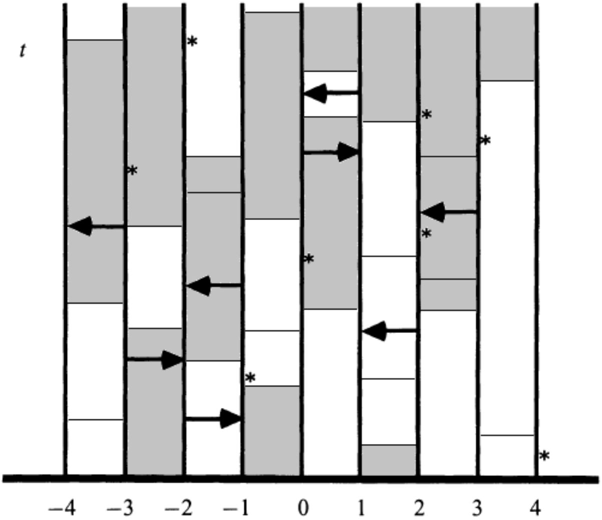
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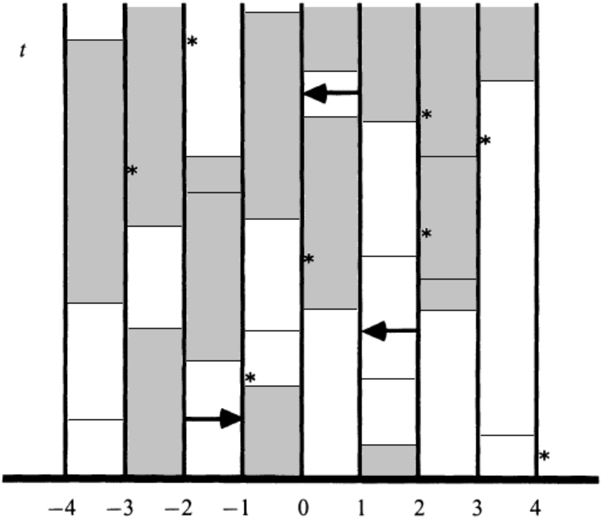
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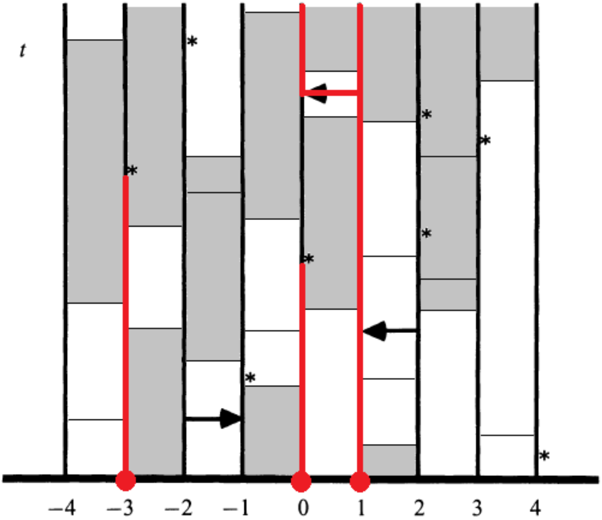
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Why dynamical bond percolation?

- Natural dynamics for Ërdos-Rényi graphs
- Simple model for the environment, broadly studied
- The geometry of the graph evolves in a smooth way
- The effect of v on the survival probability is not at all clear

Theorem

- If $\lambda < \lambda_0(\mathbb{Z})$ the process dies out
- If

$$\underline{\lambda} \equiv \frac{1}{2}\left[\lambda + v - \sqrt{(\lambda - v)^2 + 4v(1-p)\lambda}\right] \; > \; \lambda_0(\mathbb{Z})$$

the process survives

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In particular:

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- If $\lambda p > \lambda_0(\mathbb{Z})$ the process survives for big enough v

Theorem

Let η_t be the CPDP on $\mathbb{Z}$ with parameters $p \in (0,1)$, λ, and v.

For all p, λ there exist $0 < v_0 < \infty$ such that for all $v < v_0$ the process dies out.

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Corollary

For all $p \in (0,1)$ and all

$$\lambda > \lambda_0(\mathbb{Z})/p$$

the process:

- dies out if v is small enough
- survives if v is big enough

Sketch proof

For some $\epsilon > 0$ we can select $v > 0$, $k_0 \in \mathbb{N}$ and $T > 0$ so that:

- A contact process confined to $[[0, 2k_0]]$ is alive at time T with probability at most ϵ
- With probability at least $1 - \epsilon$ at least one of k_0 edges is closed during $[0, T]$ in both directions.

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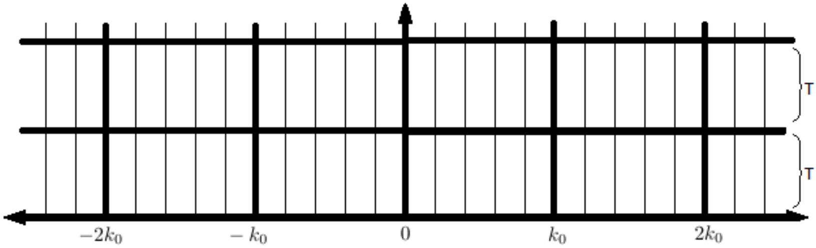

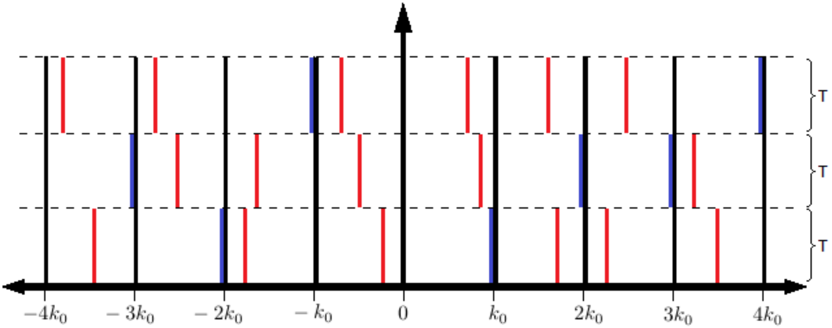
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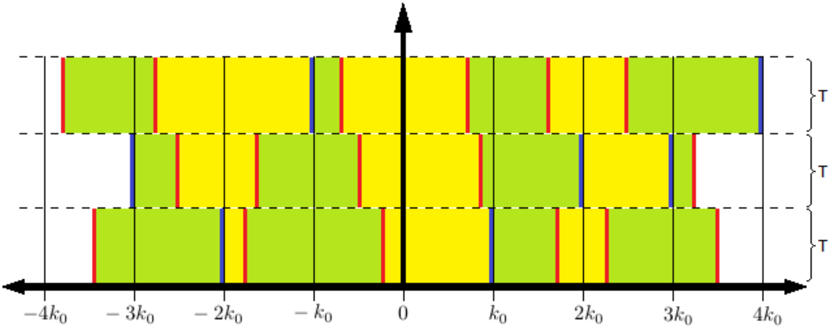
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The case of small v

Proposition

There is $p_0 > 0$ such that for all $p < p_0$ and $v > 0$ small enough, $\lambda_0 = \infty$.

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Conjecture

There is $p_1 < 1$ such that for all $p > p_1$ and $v > 0$, $\lambda_0 < \infty$.

One-dimensional long range percolation graph

- $S = \mathbb{Z}$ and $\forall x, y \in \mathbb{Z}$, $(x, y) \in E$ with probability

$$p_{|x-y|} \equiv \beta |x - y|^{-s}$$

independently of other edges.

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independently of other edges.

- $1 < s < 2 \longrightarrow$ locally finite and infinite dimension
- $s = 2 \longrightarrow$ finite dimension, dependent on β
- $s > 2 \longrightarrow$ one-dimensional with unbounded degree

Define the environment process composed of independent Markov processes $\xi_t(x,y)$ with

$$\begin{array}{ccccc} 0 & \to & 1 & \text{at rate} & \rho v_k p_k \\ 1 & \to & 0 & \text{at rate} & \rho v_k (1-p_k) \end{array}$$

where $k = |y - x|$.

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where $k = |y-x|$.

We assume that:

- $v_k p_k \to 0$
- $v_k \to \infty$
- $\sup p_k < 1$

Theorem

Under the following assumptions:

- $\sum k v_k^{-1} < \infty$
- $\sum k v_k p_k < \infty$
- $\sup p_k < 1$

there exists λ_0 such that for all $\lambda > \lambda_0$ there are

$$0 < \rho_1(\lambda) < \rho_2(\lambda) < \infty$$

such that:

- If $\rho < \rho_1$, the process dies out.
- If $\rho > \rho_2$, the process survives.