Cube Root Asymptotics for Hammersley's Model

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LPP limit fluctuations

The Hammersley Last-Passage Percolation (LPP) model was introduced by Aldous and Diaconis (95), and based on a particle system approach, developed by Hammersley (72), to study the famous Ulam's problem on the longest increasing subsequence of a random permutation.

LPP limit fluctuations

Let $\mathcal{P}$ denote a two-dimensional homogeneous Poisson process and define and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, with $\mathbf{x} \leq \mathbf{y}$ (coordinate-wise),

$$\Pi(\mathbf{x}, \mathbf{y}) := \{ \text{non-decreasing Poisson-paths } \pi \text{ in } (\mathbf{x}, \mathbf{y}] \} .$$

and

$$L(\mathbf{x}, \mathbf{y}) := \max_{\pi \in \Pi(\mathbf{x}, \mathbf{y})} \# \pi .$$

We denote $L(\mathbf{x}) := L(\mathbf{0}, \mathbf{x})$.

LPP limit fluctuations

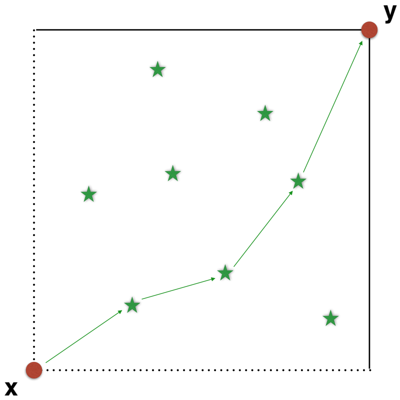

Figure: $L(\mathbf{x}, \mathbf{y}) = 3$

LPP limit fluctuations

Shape Theorem (Aldous and Diaconnis (95))

$$\lim_{n\to\infty} \frac{L(nx, nt)}{n} \stackrel{a.s.}{=} 2\sqrt{xt}\,.$$

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Limit Fluctuations (Baik, Deif and Johansson (99))

$$\lim_{n\to\infty} \frac{L(n, n) - 2n}{n^{1/3}} \stackrel{dist.}{=} \mathrm{TW}\,(\mathrm{GUE})\,.$$

LPP limit fluctuations

The Airy Process (Prähofer and Spohn (02))

For $x \in \mathbb{R}$ denote $[x]_n := (n + 2xn^{2/3}, n)$. Define the process

$$x \in \mathbb{R} \mapsto A_n(x) := \frac{L([x]_n) - (2n + 2xn^{2/3})}{n^{1/3}} + x^2 .$$

Then

$$\lim_{n\to\infty} A_n(\cdot) \stackrel{dist.}{=} A(\cdot) ,$$

where A is the Airy process. This process describes the limit fluctuations of many models and it gives rise to a universality class named the Kadar-Parisi-Zhang (KPZ) universality class.

LPP limit fluctuations

KPZ fixed point (Corwin, Quastel, Remenik (15))

It is conjectured that this universality class has space and time fluctuations described in terms of a class of evolution operators $(T_{s,t}\,,\ 0 \le s < t)$, acting on real functions, and defined as

$$T_{s,t}f(x) := \sup_{z\in\mathbb{R}} \left\{ f(z) + A(z,s;x,t) - \frac{(z-x)^2}{t-s} \right\} .$$

The (four-dimensional) process

$$\left(A(z,s;x,t)\,,\ z,x \in \mathbb{R}\,,\ 0 \le s < t \right)$$

is the so called space-time Airy sheet.

LPP limit fluctuations

LPP limit fluctuations

For $x \in \mathbb{R}$ and $t \geq$ denote $[x,t]_n := (nt + 2xn^{2/3}, nt)$. Define the space-time process A_n as

$$A_n(z,s;x,t) := \frac{L([z,s]_n,[x,t]_n) - \gamma_n(z,s;x,t)}{n^{1/3}} + \frac{(x-z)}{t-s}^2 .$$

where

$$\gamma_n(z,s;x,t) := 2(t-s)n + 2(x-z)n^{2/3} .$$

It is conjectured that

$$\lim_{n\to\infty} A_n(z,s;x,t) \stackrel{dist.}{=} A(z,s;x,t) \ \text{(as a space-time process)} .$$

LPP limit fluctuations

Goal

- Tightness of the limit fluctuations (in space);
- Gaussian local behaviour of the limit process.

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Technique

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The method of proof is base on an equilibrium comparison technique. A remarkable aspect of this method is that it gives us tightness, continuity and Gaussianity in a very directed way. The main referencs are Cator and Groeneboom (06), Cator and Pimentel (15).

Hammersley's model

Stationary model (LPP with Poisson boundary)

Let ν_λ be a homogeneous one-dimensional Poisson process (independent of $\mathcal{P}$). For $t \geq 0$ and $x \in \mathbb{R}$ define

$$L_\lambda(x,t) := \max_{z \leq x} \{ \nu_\lambda(z) + L((z,0),(x,t)) \} \ ,$$

where

$$\nu_\lambda(z) = \begin{cases} \nu_\lambda((0,z]) & \text{for } z > 0 \\ -\nu_\lambda((z,0]) & \text{for } z \leq 0 \, . \end{cases}$$

Then, for all $t \geq 0$,

$$L_\lambda(x,t) - L_\lambda(z,t) \stackrel{dist.}{=} \nu_\lambda((z,x]) \, .$$

Hammersley's model

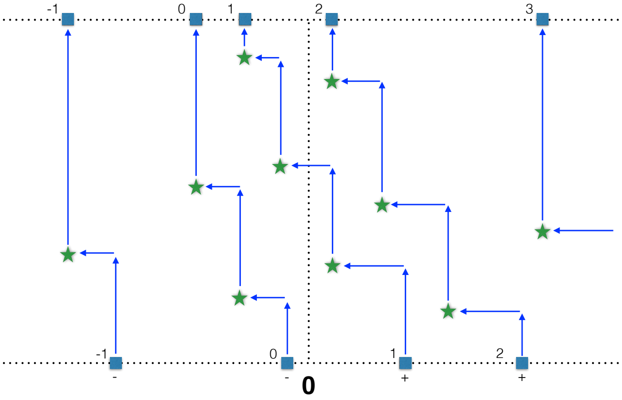

Figure: Stationary model

Exit-point formula

Theorem 1 (Cator and Groeneboom (06))

Denote $z^+ := \max\{0, z\}$, and let

$$Z_\lambda(x,t) := \sup\left\{z \le x \,:\, L_\lambda(x,t) = \nu_\lambda(z) + L((z,0),(x,t))\right\},$$

Then

$$\mathrm{Var} L_\lambda(x,t) = \frac{t}{\lambda} - \lambda x + 2\lambda \mathbb{E} Z_1(x,t)^+ .$$

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Proof of Theorem 1

- Lemma 1: $\mathrm{Var} L_\lambda(x,t) = \frac{t}{\lambda} - \lambda x + 2\mathrm{Cov}(L_\lambda(x,t), \nu_\lambda(x))$.
- Lemma 2: $\mathrm{Cov}(L_\lambda(x,t), \nu_\lambda(x)) = \lambda \mathbb{E} Z_1(x,t)^+$.

Cube root asymptotics

Theorem 2 (Cator and Groeneboom (06))

There exists $c_1 > 0$ such that, for all $t, u > 0$

$$\mathbb{P}\left(Z_1(t,t) \geq u\right) \leq c_1 t^2 \left(\frac{1}{u^3} + \frac{\mathbb{E} Z_1(t,t)^+}{u^4}\right) .$$

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Corollary 1

- $\limsup_{t\to\infty} \frac{\mathrm{Var} L_1(t,t)}{t^{2/3}} = 2 \limsup_{t\to\infty} \frac{\mathbb{E} Z_1(t,t)^+}{t^{2/3}} < \infty$;

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- There exists $c_2 > 0$ such that, for all $t, r > 0$,

$$\mathbb{P}\left(Z_1(t,t) \geq r t^{2/3}\right) \leq \frac{c_2}{r^3}.$$

Cube root asymptotics

Proof of Theorem 2

- Let $\lambda > 1$, then

$$\mathbb{P}\left(Z_1(t,t) > u\right) \leq \mathbb{P}\left(\nu_\lambda(u) - \nu_1(u) \leq L_\lambda(t,t) - L_1(t,t)\right) ;$$

Cube root asymptotics

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- For $u \in (0, \frac{3}{4}t]$, choose

$$\lambda = \lambda(u,t) := (1 - u/t)^{-1/2} ;$$

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- For $u \in (0, \frac{3}{4}t]$, choose

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- Use Chebyshev and Theorem 1.

Cube root asymptotics

Remark 1

A lower bound with the same scaling exponent can be obtained by using a similar approach. However, we will not tackle this issue here.

Cube root asymptotics

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Theorem 3

There exists $c_3 > 0$ such that, for all $t > 0$ and $r > 1$,

$$\mathbb{P}\left(|L_1(t,t) - L(t,t)| \geq rt^{1/3}\right) \leq \frac{c_3}{r^{5/4}} .$$

Cube root asymptotics

Lemma 3

Let $x < y$. If $Z_\lambda(y,t) \le 0$ then

$$L(y,t) - L(x,t) \ge L_\lambda(y,t) - L_\lambda(x,t)\,,$$

and if $Z_\lambda(x,t) \ge 0$ then

$$L(y,t) - L(x,t) \le L_\lambda(y,t) - L_\lambda(x,t)\,.$$

Proof of Lemma 3

Denote $z_y := Z_\lambda(y,t) \le 0$ and take $\mathbf{z}$ as the intersection between a maximal path from $(z_y, 0)$ to (y,t) and a maximal path from $\mathbf{0}$ to (x,t). Then

$$L_\lambda(y,t) - L_\lambda(x,t) \le L(\mathbf{z},(y,t)) - L(\mathbf{z},(x,t)) \le L(y,t) - L(x,t)\,.$$

Cube root asymptotics

Lemma 4

For $t > 0$ and $r \in [1, t^{1/3})$ set

$$\lambda_{\pm} := 1 \pm rt^{-1/3} .$$

Then there exists $c_4 > 0$ such that (for large t)

$$\mathbb{P}\left(Z_{\lambda-}(t,t) > 0\right) \leq c_4 r^{-3} ,$$

and

$$\mathbb{P}\left(Z_{\lambda+}(t,t) < 0\right) \leq c_4 r^{-3} .$$

Cube root asymptotics

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$$\lambda_{\pm} := 1 \pm r t^{-1/3}.$$

Then there exists $c_4 > 0$ such that (for large t)

$$\mathbb{P}(Z_{\lambda_-}(t,t) > 0) \leq c_4 r^{-3},$$

and

$$\mathbb{P}(Z_{\lambda_+}(t,t) < 0) \leq c_4 r^{-3}.$$

Proof of Lemma 4

Notice that

$$\begin{aligned} \mathbb{P}(Z_{\lambda_-}(t,t) > 0) &= \mathbb{P}\left(Z_1(\lambda_- t, \lambda_-^{-1} t) > 0\right) \\ &= \mathbb{P}\left(Z_1(\lambda_-^{-1} t, \lambda_-^{-1} t) > (\lambda_-^{-1} - \lambda_-) t\right). \end{aligned}$$

Cube root asymptotics

Proof of Theorem 3

Lemmas 3 and 4 are the key tool to handle the local equilibrium comparison method.

- Write $L_1(t,t) - L(t,t) = \max_{z \leq t}\{\nu_1(z) - \Delta(z,t)\}$ where $\Delta(z,t) := L(t,t) - L((z,0),(t,t))$.
- Use local comparison to show that (for $r \in [1, t^{5/4}))$

$$I(t,r) := \mathbb{P}\left(\max_{|z| \leq r^{7/12} t^{2/3}} \{\nu_1(z) - \Delta(z,t)\} > rt^{1/3}\right) \leq c_5 r^{-5/4}\,.$$

- Apply Corollary 1:

$$\begin{aligned}\mathbb{P}\left(|L_1(t,t) - L(t,t)| \geq rt^{1/3}\right) &\leq I(t,r) \\ &+ \mathbb{P}\left(|Z_1(t,t)| > r^{7/12} t^{2/3}\right)\,.\end{aligned}$$

Tightness and Gaussianity

As a consequence of Corollary 1 and Theorem 3, we have that $A_n(\,0\,,0;\;0\,,1)$ is tight. To get tightness we now need to study the modulus of continuity of

$$A_n(z,x) := A_n(\,z\,,0;\;x\,,1)\,.$$

We will do it in the one-dimensional case, by taking $z = 0$, but we hope that it will be clear how to proceed in the two-dimensional setting.

Tightness and Gaussianity

Theorem 4

Denote $A_n(x) := A_n(0, x)$. Let $\eta, \delta > 0$. There exists a constant $c_4 > 0$ such that, if with $\delta < (\eta/8)^{1/(1-\beta)}$ then

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{y\in[x,x+\delta]} |A_n(y) - A_n(x)| > \eta \right) \leq 2\mathbb{P}\left(\sup_{v\in[0,1]} |B(v)| > \frac{\eta}{2\sqrt{2\delta}} \right) + c_4\delta^{3\beta},$$

where B is a standard Brownian Motion, and $\beta \in (1/3, 1)$ is a fixed number.

Tightness and Gaussianity

Proof of Theorem 4

For $\lambda_- < \lambda_+$ let

$$B_{n,\pm}(x) := \frac{L_{\lambda\pm}([x]_n) - L_{\lambda\pm}([0]_n) - \lambda_\pm 2xn^{2/3}}{n^{1/3}} .$$

We will choose

$$\lambda_\pm = \lambda_\pm(n,\delta) := 1 \pm \frac{\delta^{-\beta}}{n^{1/3}} ,$$

where $\delta > 0$ and $\beta \in (1/3, 1)$ is a fixed number. (Later we will send $n \to \infty$ and $\delta \to 0$.) Define the event

$$E_n(\delta) := \left\{ Z_{\lambda_+}([0]_n) \geq 0 \;\text{ and }\; Z_{\lambda_-}([1]_n) \leq 0 \right\} .$$

Tightness and Gaussianity

Proof of Theorem 4

By Lemma 3, on the event $E_n(\delta)$, and for $y \in [x, x+\delta]$,

$$A_n(y) - A_n(x) \geq B_{n,-}(y) - B_{n,-}(x) - 2\delta^{1-\beta}$$

and

$$A_n(y) - A_n(x) \leq B_{n,+}(y) - B_{n,+}(x) + 4\delta^{1-\beta}\,.$$

By Lemma 4,

$$\limsup_{n\to\infty} \mathbb{P}\left(E_n(\delta)^c\right) \leq 2c_4\delta^{3\beta}\,.$$

Since $B_{n,\pm}$ are normalized Poisson processes, and $\lambda_\pm \to 1$, we have convergence to Brownian Motion (with $\sigma = 2$).

Tightness and Gaussianity

Theorem 5

Define A^ϵ by

$$A^\epsilon(u) := \epsilon^{-1/2}\left(A(\epsilon u) - A(0)\right) .$$

Then

$$\lim_{\epsilon\to 0} A^\epsilon(\cdot) \overset{dist.}{=} \sqrt{2}B(\cdot) ,$$

Proof of Theorem 5

Now choose (for fixed $\beta \in (0, 1/2)$)

$$\lambda_\pm = \lambda_\pm(n,\epsilon) := 1 \pm \frac{\epsilon^{-\beta}}{n^{1/3}} ,$$

and denote

$$A_n^\epsilon(u) := \epsilon^{-1/2}\left(A_n(\epsilon u) - A_n(0)\right) \;\text{ and }\; B_{n,\pm}^\epsilon(u) := \epsilon^{-1/2} B_{n,\pm}(\epsilon u) .$$

Tightness and Gaussianity

Proof of Theorem 5

By Lemma 3, on the event $E_n(\epsilon)$, and for $v \in [u, u+\delta]$,

$$A_n^\epsilon(v) - A_n^\epsilon(u) \geq B_{n,-}^\epsilon(v) - B_{n,-}^\epsilon(u) - 2\delta\epsilon^{1/2-\beta}$$

and

$$A_n^\epsilon(v) - A_n^\epsilon(u) \leq B_{n,+}^\epsilon(v) - B_{n,+}^\epsilon(u) + 4\delta\epsilon^{1/2-\beta}\,.$$

By using the convergence of A_n to A, together with Lemma 4,

$$\begin{aligned}\limsup_{\epsilon\to 0^+} \mathbb{P}\left(\sup_{v\in[u,u+\delta]} |A^\epsilon(v) - A^\epsilon(u)| > \eta\right) \\ \leq \quad 2\mathbb{P}\left(\sup_{v\in[0,1]} |B(v)| > \frac{\eta}{\sqrt{2\delta}}\right)\,.\end{aligned}$$

Tightness and Gaussianity

Proof of Theorem 5

To show finite-dimensional convergence,

$$\begin{aligned}\mathbb{P}\left(\cap_{i=1}^{k}\{A^{\epsilon}(u_i)\le a_i\}\right) &\le \mathbb{P}\left(\cap_{i=1}^{k}\left\{B(2u_i)\le a_i+4\epsilon^{1/2-\beta}\right\}\right)\\ &+ c_4\epsilon^{3\beta},\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(\cap_{i=1}^{k}\{A^{\epsilon}(u_i)\le a_i\}\right) &\ge \mathbb{P}\left(\cap_{i=1}^{k}\left\{B(2u_i)\le a_i-4\epsilon^{1/2-\beta}\right\}\right)\\ &- c_4\epsilon^{3\beta},\end{aligned}$$

(again by local comparison).

Tightness and Gaussianity

Remark 2

If one looks at a window of size n^γ, than the local fluctuations (rescaled by $n^{\gamma/2}$) will be Gaussian. To get that, one can parallel the local comparison argument as before, but now choosing the intensity parameter as

$$\lambda_\pm := 1 \pm \frac{1}{n^{\gamma'/2}}\,.$$

where $\gamma' \in (\gamma, 2/3)$.

Tightness and Gaussianity

Remark 3

Tightness of the sequence $A_n(\cdot,\cdot)$ show existence of a collection of Airy Sheets (only in the space varibles), though weak limits of $A_n(\cdot,\cdot)$. The finite dimensional distributions are unknown. However, the local comparison method can be applied to show that it (any weak limit) is locally an additive Brownian motion.