Branching processes in random environment.

Juan Carlos Pardo

CIMAT, Mexico

Progamme of the short course:

- Galton-Watson (GW) processes
- GW processes in random environment (GWRE).
- Scaling limits of GWRE
- CB processes in random environment.

Galton-Watson processes

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Moreover, we introduce $\{X_{i,n}; n \geq 0, i \geq 1\}$ an i.i.d. sequence of r.v.'s where each $X_{i,n}$ represents the number of offspring of the i-th individual at generation n.

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In other words, Z_{n+1} can be written as follows:

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n}.$$

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Let us denote by $\mathbb{P}$ for the law of the process. Thus,

$$\mathbb{P}\Big(X_{i,n} = k\Big) = \rho_k \qquad \text{and} \qquad \sum_{k=0}^{\infty} \rho_k = 1.$$

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Moreover, the process $(Z_n, \geq 0)$ is a Markov chain whose transition probabilities are given

$$\mathbf{P}_{ij} = \mathbb{P}\Big(Z_{n+1} = j | Z_n = i\Big) = \mathbb{P}\left(\sum_{k=1}^{i} X_{k,n} = j\right).$$

Now, we introduce the moment-generating function of Z_n

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Moreover, a straightforward computation allow us to deduce

$$\begin{aligned} f_{n+1}(s) &= \sum_{k=0}^{\infty} \mathbb{P}(Z_{n+1} = k)s^k = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{P}(Z_{n+1} = k | Z_n = j)\mathbb{P}(Z_n = j)s^k \\ &= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\mathbb{E}\left[s^{\sum_{i=1}^{j} X_{i,n}}\right] = \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\left(\mathbb{E}[s^{X_{1,0}}]\right)^j \\ &= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\left[f(s)\right]^j = f_n(f(s)). \end{aligned}$$

The latter implies

$$f_{n+1}(s) = f_{n-k}(f_{k+1}(s)) \qquad \text{for} \quad k \in \{0, 1, 2, \ldots, n-1\}.$$

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As a consecuence, we can compute the mean and variance of the process at time n, i.e. let $\mu = \mathbb{E}[X_1] < \infty$ and $\sigma^2 = \mathrm{Var}[X_1] < \infty$, then

$$\mathbb{E}[Z_n] = \mu^n \qquad \text{y} \qquad \mathrm{Var}[Z_n] = \begin{cases} n\sigma^2 & \text{if } \mu = 1, \\ \sigma^2 \mu^{(n-1)} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1. \end{cases}$$

Observe that accordingly as $\mu < 1$, $\mu = 1$ or $\mu > 1$, the expectation $\mathbb{E}[Z_n]$ decrease, is a constant or increases. Respectively, we say that the process Z is subcritical, critical or supercritical.

The previous iteration also help us to compute the extinction probability of the process Z. Let us denote

$$\{\mathrm{Ext}\} = \{\text{there is } n : Z_n = 0\},$$

and $\eta = \mathbb{P}(\mathrm{Ext})$.

Theorem

If extinction does not occur, then $\lim_{n\to\infty} Z_n = +\infty$. Moreover, we have

$$\lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \mathbb{P}(Ext) = \eta,$$

and η is the smallest non-negative root of $s = f_1(s)$. In particular, $\eta = 1$ if $\mu \leq 1$ and $\eta < 1$ if $\mu > 1$ whenever $\mathbb{P}(Z_1 = 1) < 1$.

Idea of the proof: Oserve $f_n(0) = \mathbb{P}(Z_n = 0)$. Since $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$ and therefore

$$\eta = \mathbb{P}\left(\bigcup_{n\geq 1}\{Z_n = 0\}\right) = \lim_{n\to\infty}\mathbb{P}(Z_n = 0) = \lim_{n\to\infty} f_n(0).$$

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Since $f_n(0) = f(f_{n-1}(0))$, and $\lim_{n\to\infty} f_n(0) = \eta$, from the dominated convergence Theorem, we deduce

$$\eta = \lim_{n\to\infty} f(f_{n-1}(0)) = \lim_{n\to\infty} \mathbb{E}\Big[(f_{n-1}(0))^{Z_1}\Big] = \mathbb{E}[\eta^{Z_1}] = f(\eta).$$

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The rest of the proof follows from the shape of $f_1(s)$.

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Theorem

Assume that $1 < m < \infty$. Then,

$$\mathbb{E}[W] = 1 \Leftrightarrow \mathbb{P}(W > 0 | non - extinction) = 1 \Leftrightarrow \mathbb{E}[Z_1 \log^+ Z_1] < \infty.$$

GW processes in random environment

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Let $\mathbf{e} = (\mathbf{Q}_i)_{i \geq 0}$ be a sequences of i.i.d. r.v.'s taking values in Δ. Conditioned on $\mathbf{e}$ the GWRE $(Z_i)_{i \geq 0}$ is defined as

$$Z_{i+1} = \sum_{j=1}^{Z_i} X_{j,i}, \qquad i \geq 0,$$

where Z_0 is independent of $\mathbf{e}$ and where $\{X_{j,i}; j, i \geq 0\}$ conditioned on $\mathbf{e}$ are i.i.d. with common distribution

$$\mathbb{P}\left(X_{j,i} = k \middle| \mathbf{e}\right) = \mathbf{Q}_i(k), \quad j, i, k \geq 0.$$

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We assume that $\log \mathbf{m}_0$ is a.s. finite and see that the conditional expectation of Z_i, given the environment $\mathbf{e}$, satisfies

$$\mu_i := \mathbb{E}\left[Z_i \middle| Z_0, \mathbf{e} \right] = Z_0 \prod_{k=0}^{i} \mathbf{m}_k = Z_0 e^{S_i}.$$

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iii) If the random walk S oscillates, i.e.

$$\limsup_{k\to\infty} S_k = \infty \qquad \text{and} \qquad \liminf_{k\to\infty} S_k = -\infty, \quad \text{a.s.}$$

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Observe that the estimate

$$\begin{aligned}\mathbb{P}(Z_i > 0 | Z_0, \mathbf{e}) &= \min_{0 \le k \le i} \mathbb{P}(Z_k > 0 | Z_0, \mathbf{e}) \\ &\le \min_{0 \le k \le i} \mathbb{E}[Z_k | Z_0, \mathbf{e}] \\ &= Z_0 \, e^{\min_{0 \le k \le i} S_k},\end{aligned}$$

implies that $\mathbb{P}(Z_i > 0 | Z_0, \mathbf{e})$ goes to 0 in the critical and subcritical cases, and consecuently

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As was observed by Afanasyev (80), and later independently by Dekking (88), there are three possibilities for the asymptotic behavior of subcritical branching processes. These regimes are called as weakly subcritical, intermediately subcritical and strongly subcritical.

Theorem (Kozlov, 76)

For a critical GWRE whose random environment satisfies some moment conditions, then there are some constants $0 < c_1 \leq c_2 < \infty$ *such that* $n \geq 1$

$$c_1 n^{-1/2} \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2}.$$

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Theorem (Afanasyev et al., 05)

For a critical GWRE whose random environment satisfy some moment conditions and that there exist $0 < \rho < 1$ such that

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{P}(S_m > 0) \to \rho, \qquad as \quad n \to \infty.$$

Then there is a positive constant θ such that

$$\mathbb{P}(Z_n > 0) \sim \theta n^{-(1-\rho)} l(n), \qquad as \quad n \to \infty,$$

where $(l(n))_{n\geq 1}$ is a slowly varying sequence at infinity.

Theorem (Guivarc'h and Liu, 01)

For a subcritical GWRE, we have

i) *If* $\mathbb{E}[m_0 \log m_0] < 0$ *and* $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$, *then for some constant* $c \in (0, \infty)$

$$\mathbb{P}(Z_n > 0) \sim c(\mathbb{E}[Z_1])^n, \qquad as \quad n \to \infty,$$

ii) *If* $\mathbb{E}[m_0 \log m_0] = 0, \mathbb{P}(m_0 = 1) < 1$ *and* $\mathbb{E}[m_0^2] < \infty$, *then for some constant* $0 < c_1 \leq c_2 < \infty$

$$c_1 n^{-1/2}(\mathbb{E}[Z_1])^n \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2}(\mathbb{E}[Z_1])^n \qquad as \quad n \to \infty,$$

iii) *If* $\mathbb{E}[m_0 \log m_0] > 0$ *and* $\mathbb{E}[m_0^2] < \infty$, *then for some constant* $0 < c_3 \leq c_4 < \infty$

$$c_3 n^{-3/2}\rho^n \leq \mathbb{P}(Z_n > 0) \leq c_4 n^{-3/2}\rho^n \qquad as \quad n \to \infty,$$

where $\rho = \inf_{0 \leq t \leq 1} \mathbb{E}[m_0^t]$

Theorem (Guivarc'h and Liu, 01)

Let $p > 1$, fixed and $(Z_n, n \geq 0)$ be a supercritical GWRE. Then, the following assertions are equivalent:

i) $0 < \mathbb{E}[W^p] < \infty$

ii) $\mathbb{E}[m_0^{-(p-1)}] < 1$ *and* $\mathbb{E}[(Z_1/m_0)^p] < \infty$.

iii)

$$W_n = \frac{Z_n}{\prod_{i=0}^{n} m_i} \to W \qquad in\ L^p.$$

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$$Z_{i+1}^{(n)} = \sum_{j=1}^{Z_i^{(n)}} X_{j,i}, \qquad i \geq 0,$$

where $Z_0^{(n)}$ is independent of $\mathbf{e}^{(n)}$ and where $(X_{j,i}^{(n)})_{j,i\geq 0}$ conditioned on $\mathbf{e}^{(n)}$ are i.i.d. with common distribution

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Again, let

$$\mathbf{m}_i^{(n)} = \sum_{k\geq 0} k\mathbf{Q}_i^{(n)}(k).$$

Let $Z_t^{(n)} = Z_{[t]}^{(n)}$ be a continuous time version of the GWRE and $(S_t^{(n)})_{t\geq 0}$ be its associated random walk which is defined by

$$S_t^{(n)} = \sqrt{n} \sum_{i=i}^{[t]-1} \log\left(\mathbf{m}_i^{(n)}\right), \qquad t \geq 0.$$

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$$\lim_{n\to\infty} n\mathbb{E}[(m_1^{(n)} - 1)] = \beta \in \mathbb{R}.$$

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$$\sup_{n\geq 0} \mathbb{E}\left[\sum_{k\geq 0} \left|\frac{k}{m_1^{(n)}} - 1\right|^2 \mathbf{Q}_1^{(n)}(k)\right] < \infty.$$

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we have (Kurtz, 78)

$$\left(\frac{Z_{tn}^{(n)}}{n},\frac{S_{tn}^{(n)}}{\sqrt{n}}\right)_{t\geq 0}\to (Z_t,S_t)_{t\geq 0}$$

in the sense of Skorokhod.

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in the sense of Skorokhod. Moreover

$$Z_t=Z_0+\frac{\sigma^2}{2}\int_0^t Z_s\,ds+\int_0^t\sqrt{2\gamma^2 Z_s}\,dB_s+\int_0^t Z_s\,dS_s,$$

where $S_t=\beta t+\sigma W_t, t\geq 0$, $\beta\in\mathbb{R}$ and B and W are two independent standard Brownian motions.

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