Branching processes in random environment.

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Branching diffusions in a random environment

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$$Z_{i+1}^{(n)} = \sum_{j=1}^{Z_i^{(n)}} X_{j,i}, \qquad i \geq 0,$$

where $Z_0^{(n)}$ is independent of $\mathbf{e}^{(n)}$ and where $(X_{j,i}^{(n)})_{j,i\geq 0}$ conditioned on $\mathbf{e}^{(n)}$ are i.i.d. with common distribution

$$\mathbb{P}\left(X_{j,i}^{(n)} = k \middle| \mathbf{e}^{(n)}\right) = \mathbf{Q}_i^{(n)}(k), \quad j, i, k \geq 0.$$

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Let

$$\mathbf{m}_i^{(n)} = \sum_{k\geq 0} k\mathbf{Q}_i^{(n)}(k).$$

Let $Z_t^{(n)} = Z_{[t]}^{(n)}$ be a continuous time version of the GWRE and $(S_t^{(n)})_{t\geq 0}$ be its associated random walk which is defined by

$$S_t^{(n)} = \sqrt{n} \sum_{i=i}^{[t]-1} \log\left(\mathbf{m}_i^{(n)}\right), \qquad t \geq 0.$$

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If $Z_0^{(n)}/n \to Z_0$ in law. Hence under some *technical* assumptions we have

$$\left(\frac{Z_{tn}^{(n)}}{n}, \frac{S_{tn}^{(n)}}{\sqrt{n}}\right)_{t\geq 0} \to (Z_t, S_t)_{t\geq 0}$$

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$$\left(\frac{Z_{tn}^{(n)}}{n}, \frac{S_{tn}^{(n)}}{\sqrt{n}}\right)_{t\geq 0} \to (Z_t, S_t)_{t\geq 0}$$

in the sense of Skorokhod. Moreover

$$Z_t = Z_0 + a\int_0^t Z_s\, ds + \int_0^t \sqrt{2\gamma^2 Z_s}\, dB_s + \int_0^t Z_s\, dS_s,$$

where $S_t = \beta t + \sigma W_t$, $t \geq 0$, $\beta \in \mathbb{R}$ and B and W are two independent standard Brownian motions.

We are interested on the strong solutions of

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For the moment, we assume that there is a unique strong solution of the previous SDE and for simplicity, we study the case where $\sigma = \alpha = \beta = 0$. For convenience, we denote by Y the unique strong solution of

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Since the previous SDE has a unique strong solution of the above equation, then Y is Markovian and satisfies the branching property.

Indeed, let $\mathbb{P}_x$ denotes the law of Y such that $\mathbb{P}_x(Y_0 = x) = 1$. Then we would like to show that $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

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For two independent BM's B and $\widetilde{B}$, we call Y and $\widetilde{Y}$ the corresponding two solutions with x and y and consider $X = Y + \widetilde{Y}$.

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Let β be a third BM independent of B and $\widetilde{B}$. The process

$$C_t = \int_0^t \mathbf{1}_{\{X_s > 0\}} \frac{\sqrt{2\gamma^2 Y_s} dB_s + \sqrt{2\gamma^2 \widetilde{Y}_s} d\widetilde{B}_s}{\sqrt{X_s}} + \int_0^t \mathbf{1}_{\{X_s = 0\}} d\beta_s,$$

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is a linear BM since $\langle C \rangle_t = t$, implying

$$X_t = x + y + \int_0^t 2\sqrt{\gamma^2 X_s} dC_s.$$

Take $\lambda \geq 0$. From Itô's formula, one can deduce that

$$M_s = \exp\left\{\frac{-\lambda Y_s}{1+\gamma^2\lambda(t-s)}\right\}, \qquad \text{for } 0 \leq s \leq t,$$

is a martingale.

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Therefore

$$\mathbb{E}_x[e^{-\lambda Y_t}] = \exp\left\{\frac{-\lambda x}{1+\gamma^2\lambda t}\right\}.$$

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Therefore

$$\mathbb{E}_x[e^{-\lambda Y_t}] = \exp\left\{\frac{-\lambda x}{1+\gamma^2\lambda t}\right\}.$$

Hence, the survival probability satisfies

$$\mathbb{P}_x(Y_t > 0) = 1 - e^{-x(\gamma^2 t)^{-1}}.$$

Recall that

$$Z_t = Z_0 + (a+\beta)\int_0^t Z_s\,ds + \int_0^t \sqrt{2\gamma^2 Z_s}\,dB_s + \sigma\int_0^t Z_s\,dW_s. \qquad (1)$$

and

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Proposition

Let $\mathbf{m} = \beta + a - \sigma^2/2$ *and define*

$$\tau(t) = \int_0^t e^{-\sigma W_s - \mathbf{m}s}\,ds.$$

Hence the process $(e^{\sigma W_t + \mathbf{m}t} Y_{\tau(t)}, t \geq 0)$ *is a weak solution of (1).*

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$$\left\langle e^{\sigma W_\cdot + \mathbf{m} \cdot}, Y_{\tau(\cdot)} \right\rangle_t = 0.$$

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Applying again Itô's formula and from the definition of Y as an SDE, we deduce

$$Z_t = Y_0 + \sigma \int_0^t Z_s dW_s + (\mathbf{m} + \sigma^2/2) \int_0^t Z_s ds + \int_0^t e^{\sigma W_s + \mathbf{m}s} \sqrt{2\gamma^2 Y_{\tau(s)}} dB_{\tau(s)}.$$

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The proof follows by noting that $\mathbf{m} + \sigma^2/2 = \beta + a$ and

$$M_t = \int_0^t \sqrt{e^{\sigma W_s + \mathbf{m}s}} dB_{\tau(s)}$$

is a Brownian motion. The latter holds since $\langle M \rangle_t = t$.

Corollary

Let Z be a branching diffusion in a Brownian random environment (BDRE). Then a.s.

$$\mathbb{E}_x[e^{-\lambda Z_t}|B] = \exp\left\{-x\left(\lambda^{-1} + \gamma^2 \int_0^t e^{-\sigma W_s - \mathbf{m}s} ds\right)^{-1}\right\}.$$

In particular

$$\mathbb{P}(Z_t > 0|B) = 1 - \exp\left\{-x\left(\gamma^2 \int_0^t e^{-\sigma W_s - \mathbf{m}s} ds\right)^{-1}\right\}.$$

Hence the probability of extinction depends on the law of

$$A_t^{(\eta)} := \int_0^t \exp\left\{2(\eta s + W_s)\right\} \mathrm{d}s, \qquad t \in [0, \infty).$$

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According to Matsumoto and Yor (2003), for all $t,\ u \in (0, \infty)$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}(A_t^{(\eta)} \in \mathrm{d}u | \, W_t + \eta t = x) = \frac{\sqrt{2\pi t}}{u} e^{\frac{x^2}{2t}} e^{-\frac{1}{2u}\left(1+e^{2x}\right)} \theta_{e^x/u}(t) \mathrm{d}u,$$

where θ_r is "explicit".

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According to Bertoin and Yor (05), $A_\infty^{(\eta)} = \lim_{t\to\infty} A_t^{(\eta)}$ is a finite a.s. whenever $\eta < 0$. Moreover ,

$$A_\infty^{(\eta)} \quad \text{has the same law as} \quad \Big(2\Gamma_{-\eta}\Big)^{-1},$$

where Γ_v is Gamma r.v. with parameter v.

Let $\mathbf{m} = \beta + a - \frac{\sigma^2}{2}$ and take $\eta := -\frac{2\mathbf{m}}{\sigma^2}$ and $\mathbf{k} = \frac{\sigma^2}{2\gamma^2}$.

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Theorem

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i) *(Supercritical case) If $\mathbf{m} > 0$, we have*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = 1 - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \mathbf{k}^n \frac{\Gamma(n-\eta)}{\Gamma(-\eta)}.$$

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ii) *(Critical case) If $\mathbf{m} = 0$, we have*

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = -\frac{\sqrt{2}}{\sqrt{\pi}\sigma} \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \mathbf{k}^n \Gamma(n).$$

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iii) *(Weakly subcritical) If* $\mathbf{m} \in (-\sigma^2, 0)$, *we have*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\frac{\mathbf{m}^2 t}{2\sigma^2}} \mathbb{P}_z(Z_t > 0) = \frac{8}{\sigma^3} \int_0^\infty (1 - e^{-\mathbf{k}zv}) \phi_\eta(v) \mathrm{d}v,$$

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iv) *(Intermediately subcritical) If* $\mathbf{m} = -\sigma^2$, *we have*

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v) *(Strongly subcritical) If* $\mathbf{m} < -\sigma^2$*, we have*

$$\lim_{t\to\infty} e^{\frac{\sigma^2 t}{2}} e^{-(\mathbf{m}+\sigma^2)t} \mathbb{P}_z(Z_t > 0) = z\mathbf{k}(\eta - 2).$$

Branching diffusions with catastrophes

We are now interested in the following SDE

$$Z_t = Z_0 + a\int_0^t Z_s\,ds + \int_0^t \sqrt{2\gamma^2 Z_s}\,dB_s + \int_0^t Z_{s-}\,dS_s,$$

where $a \in \mathbb{R}$ and S is a Compound Poisson process independent of B whose jump distribution is supported on $(-1, 0)$ and jumps with rate $r > 0$.

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Recall that the scaling limit of GW processes is a Feller diffusion with growth a and diffusion part γ^2. The latter is also the scaling limit of birth and death processes and it gives a natural model for populations which die and multiply fast, randomly, without interaction.

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The process Z was considered by Bansaye and Tran (2011) to model parasites growing in dividing cells.

The cell divides at constant rate r and a random fraction $\Theta \in (0, 1)$ of parasites enters the first daughter cell, whereas the remainder enters the second daughter cell. Following the infection in a cell line, the parasites grow as a Feller diffusion process and undergo a catastrophe when the cell divides.

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Observe that if there is one infected cell at time 0, $\mathbb{E}[N_t] = e^{rt}$ and $\mathbb{E}[N_t^*] = e^{rt}\,\mathbb{P}(Z_t > 0)$, since N is a Yule process.

It is important to note that the process Z can be written as follows

$$Z_t = Z_0 + a\int_0^t Z_s\,ds + \int_0^t \sqrt{2\gamma^2 Z_s}\,dB_s + \int_0^t \int_{(0,1)} (\theta - 1) Z_{s-}\,M(ds, d\theta), \quad (2)$$

where $M(ds, d\theta)$ a Poisson random measure with intensity $rds\,\mathbb{P}(\Theta \in d\theta)$.

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Proposition

Let

$$K_t = at + \int_0^t \int_{(0,1)} \log\theta\, M(ds, d\theta),$$

and define

$$\tau(t) = \int_0^t e^{-K_s}\,ds.$$

Hence the process $(e^{K_t}\,Y_{\tau(t)}, t \geq 0)$ *is a weak solution of (2).*

Proof:

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From the independence of B and M, we have $\left[e^{K_\cdot}, Y_{\tau(\cdot)}\right]_t = 0$.

By noting that

$$e^{K_t} = 1 + a \int_0^t e^{K_s} ds + \int_0^t \int_{(0,1)} e^{K_{s-}} (\theta - 1) M(ds, d\theta).$$

and Itô's formula, we deduce

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Proof: Again, we denote $Z_t = e^{K_t} Y_{\tau(t)}$. Applying Itô's formula, we deduce

$$Z_t = Y_0 + \int_0^t Y_{\tau(s)} de^{\sigma K_s} + \int_0^t e^{\sigma K_s} dY_{\tau(s)}.$$

From the independence of B and M, we have $\left[e^{K_\cdot}, Y_{\tau(\cdot)}\right]_t = 0$.

By noting that

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The proof follows by observing that

$$M_t = \int_0^t \sqrt{e^{K_s}} dB_{\tau(s)}$$

is a Brownian motion since $\langle M \rangle_t = t$.

Corollary

Let Z be a branching diffusion with catastrophes. Then a.s.

$$\mathbb{E}_x[e^{-\lambda Z_t}|K] = \exp\left\{-x\left(\lambda^{-1} + \gamma^2 \int_0^t e^{-K_s}\,ds\right)^{-1}\right\}.$$

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Unfortunately there is not a explicit form for the density of the exponential functional of K. The only case known who has explicit density is the case of Brownian motion with drift.

Assume that the Laplace exponent of the process K is well-defined for some positive real number, i.e.

$$\psi(\lambda) = \log \mathbb{E}[e^{\lambda K_1}] < \infty \quad \text{for } \lambda \in [0, \theta^+),$$

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$$\frac{1}{t} \log \mathbb{P}_x(Z_t > 0 \mid K) \to \psi'(0),$$

in probability.

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(iii) *If $\psi'(1) > 0$ (weakly s.r.) and $\theta^+ > 2$, then there exists $c_3 > 0$ such that*

$$\mathbb{P}_x(Z_t > 0) \sim c_3 t^{-3/2} e^{t\psi(\tau)+g}, \qquad as \quad t \to \infty,$$

where τ is s.t. $\psi(\tau) = \min_{0<s<1} \psi(s)$.

Proposition

b) If $\psi'(0) = 0$ (critical case) and $\theta^{+} > 1$, then for every $x > 0$, there exists $c_4 > 0$ such that

$$\mathbb{P}_x(Z_t > 0) \sim c_4 t^{-1/2}, \qquad as \quad t \to \infty.$$

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\ x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

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In particular,

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\{-xu_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some function $u_t(\lambda)$.

The function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t$$

where ψ (branching mechanism of Y) satisfies the Lévy-Khincthine formula

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

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Observe $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical accordingly as $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$.

The probability of extinction is given by

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A CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0.$$

A CB-process can also be defined as the unique non-negative strong solution of the stochastic differential equation

$$Y_t = Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s + \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Y_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure independent of B, with intensity $\mathrm{d}s \otimes \mu(\mathrm{d}z) \otimes \mathrm{d}u$ and $\tilde{N}$ is its compensated version.

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