Continuous state branching processes in a Lévy random environment.

J.C. Pardo (CIMAT)

joint work with Sandra Palau & Charline Smadi

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\, x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

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More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property.

In particular,

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some function $u_t(\lambda)$.

The function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t$$

where ψ (branching mechanism of Y) satisfies the Lévy-Khincthine formula

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

$$\int_{(0,\infty)} (1 \wedge x^2)\mu(\mathrm{d}x) < \infty.$$

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Observe $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical accordingly as $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$.

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A CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0.$$

A CB-process can also be defined as the unique non-negative strong solution of the stochastic differential equation

$$
\begin{aligned}
Y_t = & Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s \\
& + \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Y_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),
\end{aligned}
$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure independent of B, with intensity $\mathrm{d}s \otimes \mu(\mathrm{d}z) \otimes \mathrm{d}u$ and $\tilde{N}$ is its compensated version.

CB-process in a Lévy random environment

We introduce a continuous state branching process in a Lévy random environment (CBLRE) as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t = & Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s \\ & + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

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\end{aligned}
$$

where

$$
\begin{aligned}
S_t = \alpha t + \sigma B_t^{(e)} & + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) \\
& + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z),
\end{aligned}
$$

with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ its compensated version and π is a σ-finite measure satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

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We define the auxiliary process

$$K_t = \mathbf{m}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v),$$

where

$$\mathbf{m} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.

Theorem

The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and its infinitesimal generator $\mathcal{L}$ satisfies, for every $f \in C^2(\mathbb{R}_+)$,

$$\mathcal{A}f(x) = (\mathbf{m} + a)xf'(x) + \left(\gamma^2 x + \frac{\sigma^2}{2}x^2\right) f''(x)$$
$$+ x\int_{(0,\infty)} \Big(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\Big)\Lambda(\mathrm{d}\, z)$$
$$+ \int_{\mathbb{R}} \Big(f(xe^z) - f(x) - x(e^z - 1)f'(x)\mathbf{1}_{\{|z|<1\}}\Big)\pi(\mathrm{d}\, z).$$

Furthermore, the process Z, conditionally on K, satisfies the branching property and for every $t > 0$

$$\mathbb{E}_z\left[\exp\left\{ -\lambda Z_t e^{-K_t}\right\}\Big| K\right] = \exp\Big\{ -zv_t(0,\lambda,K)\Big\} \qquad a.s.,$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, D(\mathbb{R}_+))$, $v_t : s \in [0, t] \mapsto v_t(s, \lambda, \delta)$

Theorem

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$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{\delta_s} \psi(v_t(s, \lambda, \delta) e^{-\delta_s}),$$

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$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = e^{\delta_s}\psi(v_t(s,\lambda,\delta)e^{-\delta_s}),$$

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Idea of the proof: First, we fix $n \geq 1$ and prove the existence of a positive weak solution of the SDE

$$Z_t^n = Z_0^n + a\int_0^t (Z_s^n \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 (Z_s^n \wedge n)}\mathrm{d}B_s + \int_0^t (Z_s^n \wedge n)\mathrm{d}S_s \quad (1)$$

$$+ \int_0^t \int_{(0,1)} \int_0^{(Z_s^n \wedge n)-} (z\wedge n)\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{(Z_s^n \wedge n)-} (z\wedge n) N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Similar techniques as those used in Dawson & Li (2012) provides the pathwise uniqueness of (1). Basically, we take Z^n and $Z^{n,\prime}$ two solutions of (1) and prove that the expectation of $|Z^n - Z^{n,\prime}|$ equals 0.

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For $m \geq 1$ let $\tau_m = \inf\{t \geq 0 : Z_t^m \geq m\}$. By a localization argument, we may construct

$$Z_t = \begin{cases} Z_t^m & \textit{if} \quad t < \tau_m \\ \infty & \textit{if} \quad t \geq \lim\limits_{m\to\infty} \tau_m \end{cases}$$

which is a weak solution to our original equation.

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which is a weak solution to our original equation.

Finally, let Z' and Z'' be two solutions of our original equation and also consider $\tau'_m = \inf\{t \geq 0 : Z'_t \geq m\}$, $\tau''_m = \inf\{t \geq 0 : Z''_t \geq m\}$ and define $\tau_m = \tau'_m \wedge \tau''_m$.

Then, Z' and Z'' satisfy (1) on $[0, \tau_m)$, so they are indistinguishable on $[0, \tau_m)$. When $\tau_\infty = \lim_{m \to \infty} \tau_m < \infty$, Z' or Z'' have a jump of infinity size in τ_∞. This jump comes from an atom of N, so that both processes have it and thus Z' and Z'' are indistinguishable. Then, the strong solution to our original equation follows.

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The branching property of Z_t conditionally on K, is inherited from the CB-process.

Let $\widetilde{Z}_t = Z_t e^{-K_t}$ and $v_t(s, \lambda, K)$ is differentiable with respect to the variable s, non-negative and such that $v_t(t, \lambda, K) = \lambda$ for all $\lambda \geq 0$.

Applying Itô's formula one obtain that $\exp\{-\widetilde{Z}_t v_t(s,\lambda,K)\}$ conditionally on K is a martingale if and only if for every $t \geq 0$,

$$\frac{\partial}{\partial s} v_t(s,\lambda,K) = -a v_t(s,\lambda,K) + \gamma^2 (v_t(s,\lambda,K))^2 e^{-Ks}$$
$$+ e^{K_s} \int_0^\infty \left(e^{-e^{-K_s} v_t(s,\lambda,K) z} - 1 + e^{-K_s} v_t(s,\lambda,K) z \mathbf{1}_{\{z<1\}} \right) \mu(\mathrm{d}z).$$

Observe that in the case when $|\psi'(0+)| < \infty$, the auxiliary process can be taken as follows

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Using same arguments as before and replacing K with $K^{(0)}$, one can deduce that $v_t(s, \lambda, K^{(0)})$ is the unique solution to

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) = e^{K_s^{(0)}} \psi_0\big(v_t(s, \lambda, K^{(0)}) e^{-K_s^{(0)}}\big),$$

where

$$\psi_0(\lambda) = \gamma^2 \lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x\big) \mu(\mathrm{d}x).$$

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In this case, the process Z, conditionally on $K^{(0)}$, satisfies

$$\mathbb{E}_z\left[\exp\left\{ -\lambda Z_t e^{-K_t^{(0)}} \right\} \Big| K^{(0)} \right] = \exp\left\{ -z v_t(0, \lambda, K^{(0)}) \right\} \qquad a.s.$$

Examples

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Observe that $\psi'(0+) = -\infty$. In this case

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Then,

$$\mathbb{E}_z\left[\exp\left\{ -\lambda Z_t e^{-K_t} \right\} \Big| K \right] = \exp\left\{ -z\lambda^{e^{-t}} \exp\left\{ \int_0^t e^{-s} K_s \mathrm{d}s \right\} \right\} \qquad a.s.,$$

which implies that

$$\mathbb{P}_z\Big(Z_t > 0 \Big| K \Big) = 1, \qquad t > 0.$$

Observe that

$$\mathbb{E}_z\Big[Z_t e^{-K_t}\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = z e^{-t}\lambda^{e^{-t}-1} \times \exp\left\{\int_0^t e^{-s}K_s \mathrm{d}s - z\lambda^{e^{-t}}\exp\left\{\int_0^t e^{-s}K_s\mathrm{d}s\right\}\right\}.$$

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This implies

$$\mathbb{E}_z[Z_t] = \infty, \qquad t > 0.$$

Moreover, when K is just a Brownian motion with drift, the r.v. $\int_0^t e^{-s}K_s\mathrm{d}s$ is Gaussian with mean $(\alpha - \frac{\sigma^2}{2})(1 - e^{-t} - te^{-t})$ and variance $\frac{\sigma^2}{2}(1 - 4e^{-t} - 3e^{-2t})$.

Feller's diffusion

If $a = \mu(0, \infty) = 0$, the CBBRE is given by

$$Z_t = Z_0 + \alpha \int_0^t Z_s \, ds + \sigma \int_0^t Z_s \, dS_s + \int_0^t \sqrt{2\gamma^2 Z_s} \, dB_s.$$

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The above is equivalent to the strong solution of the SDE

$$\begin{aligned} \mathrm{d}Z_t =& \frac{\sigma^2}{2} Z_t \mathrm{d}t + Z_t \mathrm{d}K_t + \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s \\ \mathrm{d}K_t =& \alpha \mathrm{d}t + \sigma \mathrm{d}W_t, \end{aligned}$$

which looks as the branching diffusion in random environment studied by Böinghoff and Hutzenthaler (2011).

Stable case. Here, the branching mechanism is of the form

$$\psi(\lambda) = -a\lambda + c_\beta \lambda^{\beta+1}, \qquad \lambda \geq 0,$$

for some $\beta \in (-1,0) \cup (0,1)$, $a \in \mathbb{R}$, and

$$\begin{cases} c_\beta < 0 & \text{if } \beta \in (-1,0), \\ c_\beta > 0 & \text{if } \beta \in (0,1). \end{cases}$$

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Under this assumption, the process Z satisfies the following stochastic differential equation

$$Z_t = Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}S_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and

$$\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \begin{cases} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (-1,0), \\ \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (0,1), \end{cases}$$

where N is an independent Poisson random measure with intensity

$$\frac{c_\beta \beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}} \mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

In this case, we note

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We use in both cases the backward differential equation of Theorem 1 and observe that it satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = -a v_t(s,\lambda,\delta) + c_\beta v_t^{\beta+1}(s,\lambda,\delta) e^{-\beta\delta_s}.$$

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Therefore,

$$v_t(s,\lambda,\delta) = e^{as}\left((\lambda e^{at})^{-\beta} + \beta c_\beta \int_s^t e^{-\beta(\delta_u + au)} \mathrm{d}u \right)^{-1/\beta}.$$

Implying the following a.s. identity

$$\mathbb{E}_z\left[\exp\left\{ -\lambda Z_t e^{-K_t^{(0)}} \right\} \middle| K^{(0)} \right] = \exp\left\{ -z \left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta K_u^{(0)}} \mathrm{d}u \right)^{-1/\beta} \right\}.$$

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

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Recall that $\psi'(0+) \in [-\infty, \infty)$, and that whenever $|\psi'(0+)| < \infty$, we write

$$\mathbf{m} = \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

where

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where

$$\psi'(0+) = -a - \int_{\{x>1\}} x\mu(\mathrm{d}x).$$

Proposition

Assume $|\psi'(0+)| < \infty$, then a CBPBRE Z with branching mechanism ψ satisfies

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \textit{for all} \quad t > 0.$$

Stable case with $\beta \in (-1, 0)$.

Recall that in this case $\psi(u) = -au + c_\beta u^{\beta+1}$, where $a \in \mathbb{R}$ and c_β is a negative constant.

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From straightforward computations, we get $\psi'(0+) = -\infty$

Stable case with $\beta \in (-1, 0)$.

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From the Laplace transform of $\tilde{Z}$ (taking λ goes to 0), we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = \exp\left\{ -z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta} \right\} \qquad \text{a.s.},$$

implying

$$\mathbb{P}_z\Big(Z_t = \infty \Big| K\Big) = 1 - \exp\left\{ -z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta} \right\} > 0.$$

Moreover, if the process $(K_u + au, u \geq 0)$ does not drift to $+\infty$, we deduce from a result of Bertoin and Yor,

$$\int_0^t e^{-\beta(K_u+au)} \mathrm{d}u \to \infty, \qquad \text{as} \quad t \to \infty,$$

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On the other hand, if the process $(K_u + au, u \geq 0)$ drifts to $+\infty$, we have an interesting behaviour of the process Z. In fact, we deduce from the dominated convergence Theorem

$$\mathbb{P}_z\Big(Z_\infty = \infty\Big) = 1 - \mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta(K_u+au)} \mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

The above probability is positive since

$$\int_0^\infty e^{-\beta(K_u+au)} \mathrm{d}u < \infty \qquad \text{a.s.}$$

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By taking λ goes to 0 in the Laplace exponent of $\tilde{Z}$, one can see that the process is conservative conditionally on the environment, i.e.

$$\mathbb{P}_z(Z_t < \infty | K) = 1,$$

for all $t \in (0, \infty)$ and $z \in [0, \infty)$.

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- **i)** *If* $K^{(0)}$ *drifts to* $-\infty$, *then* $\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1$, *a.s.*
- **ii)** *If* $K^{(0)}$ *oscillates, then* $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1$, *a.s.*

  Moreover if $\gamma > 0$ *then*

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1, a.s.$$

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iii) *If $K^{(0)}$ drifts to $+\infty$ and*

$$\int^{\infty} x \ln x \, \mu(\mathrm{d}x) < \infty,$$

then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t > 0 \,\middle|\, K^{(0)}\right) > 0$ a.s., and there exists a non-negative finite r.v. W such that

$$Z_t e^{-K_t^{(0)}} \underset{t\to\infty}{\longrightarrow} W, \; a.s \qquad and \qquad \{W = 0\} = \left\{\lim_{t\to\infty} Z_t = 0\right\}.$$

Moreover if $\gamma > 0$, we have

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \geq \left(1 + \frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

It is important to note that in the Feller and stable cases, i.e. $\psi(u) = -au + c_\beta u^{\beta+1}$ for $\beta \in (0,1]$, one can deduce directly that

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The latter probability can be computed explicitly provided the Lévy process has positive exponential moments of all order.

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$$\lim_{t\to\infty} \int_0^t e^{-s} K_s \mathrm{d}s = \sigma \int_0^\infty e^{-s} dK_s + \alpha - \frac{\sigma^2}{2},$$

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exists whenever $\mathbb{E}[\log|K_1|] < \infty$. In the case where K is a Brownian motion, the latter r.v. is Gaussian with mean $\alpha - \frac{\sigma^2}{2}$ and variance $\frac{\sigma^2}{2}$. Hence,

$$\mathbb{E}_z\left[\exp\left\{-\lambda \lim_{t\to\infty} Z_t e^{-K_t}\right\}\Big| K\right] = \exp\left\{-z \exp\left\{\int_0^\infty e^{-s} K_s \mathrm{d}s\right\}\right\}.$$

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Stable case

Recall that in this case

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\}\Big|K^{(0)}\right] = \exp\left\{-z\left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta K_u^{(0)}} \mathrm{d}u\right)^{-1/\beta}\right\}.$$

Hence the probability of explosion or extinction depends on the law of

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which is a finite a.s. whenever K drifts to ∞. According to Mauzlik and Zwart ,

$$\mathbb{E}[A_\infty^s] = -\frac{\phi_K(-\beta s)}{s}\mathbb{E}[A_\infty^{s-1}],$$

for s such that $\phi_K(s)=\log\mathbb{E}[e^{sK_1}]$ is well defined.

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i) *Subcritical-explosion. If $\phi'_K(0+) < 0$, then there exist $c_1(z) > 0$ such that*

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ii) *(Critical case) If $\phi'_K(0+) = 0$ (+ some moments conditions), then there exist $c_5(z) > 0$ such that*

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iii) *(Weakly subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) > 0$ *(+ some moments conditions), then there exist* $c_6(z) > 0$ *such that*

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v) *(Strongly subcritical)If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) < 0$ *(+ some moments conditions), then there exist* $c_7 > 0$ *such that*

$$\lim_{t\to\infty} e^{t\phi_K(1)} \mathbb{P}_z(Z_t > 0) = zc_8.$$

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