Lecture notes on percolation and long-range percolation

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Lecture 1

Percolation

Simplest model: Bond-Percolation on $\mathbb{Z}^2$, *i.e.*the set of vertices $V = \mathbb{Z}^2$.

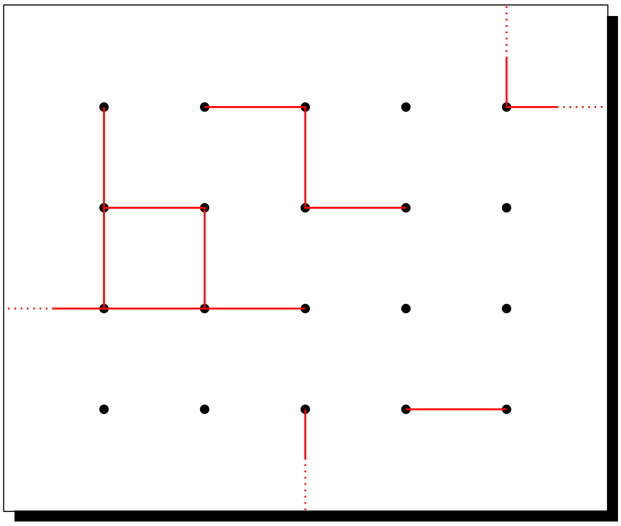


Figure 1.1: Example of Bond-Percolation on the grid $\mathbb{Z}^2$. Edges are drawn in red.

For all $v, v' \in V$ that differ in exactly one coordinate ($|v - v'|_1 = 1$), put the edge $e = \{v, v'\}$ with probability p, independently of all other edges. (See an example on Figure **??**.)
Question: Does an infinite cluster (= connected component of infinite size) exist ?
In this case, is the origin $O = (0, 0)$ contained in an infinite cluster ?
If $0 < p < 1$, $(0, 0)$ has a non-null probability $(1-p)^4$ to be an isolated vertex (and so O is not in an infinite cluster; *cf.* Figure **??**).

Define:

- $\theta(p) := \mathbb{P}\left[(0,0) \in \infty \text{ cluster}\right]$
- The critical percolation probability $p_c = \inf\{p \ : \ \theta(p) > 0\}$

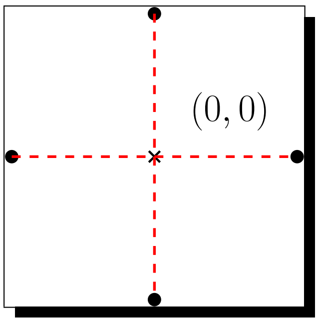


Figure 1.2: $(0, 0)$ is isolated with prob. $(1-p)^4$: the 4 hypothetical edges (red-dashed) do not appear.

Theorem: $0 < p_c < 1$ (in fact, $p_c = \frac{1}{2}$).

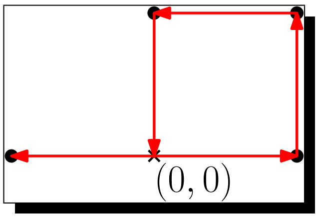


Figure 1.3: A self-avoiding path of length $n = 5$ starting at the origin.

Proof: A "self-avoiding" path of length n may not reuse edges but may reuse vertices (*cf.* example on Figure **??**). Define $\Omega_n = \{\text{self-avoiding path of length } n \text{ starting at the origin } (0,0)\}$. Then $|\Omega_n| \leq 4^n$.

Observe: $(0,0) \in \infty$ cluster $\implies \forall n$, there exists a self-avoiding path of length n.

$$\begin{aligned}\implies \theta(p) &\leq \mathbb{P}_p\left[\,\exists \text{ self-avoiding path of length } n \text{ starting at } (0,0)\,\right]\\ &\leq \sum_{\gamma\in\Omega_n} \mathbb{P}_p\left[\,\gamma \text{ is a self-avoiding path}\,\right]\\ &\leq 4^n \times p^n = (4p)^n \to 0 \text{ if } p < \frac{1}{4}\end{aligned}$$

$$\implies p_c \geq \frac{1}{4}.$$

Conversely: For the second part, look at the **dual graph** $\left(\mathbb{Z}^2\right)^* \simeq \mathbb{Z}^2$.

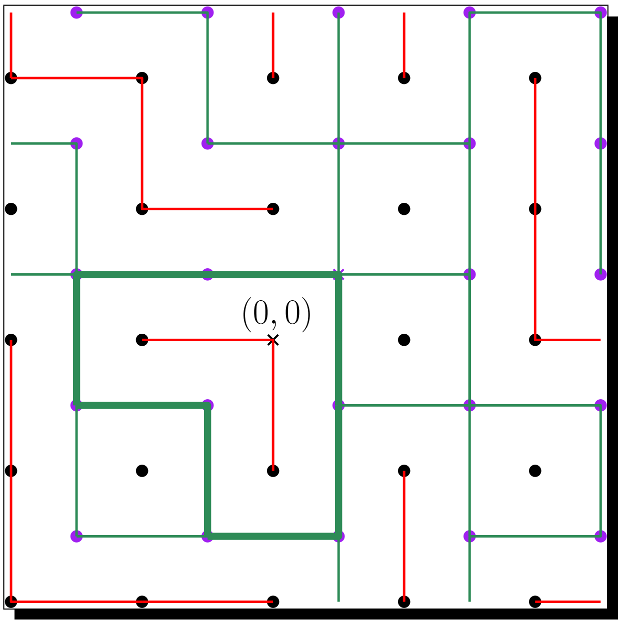


Figure 1.4: Dual graph $\left(\mathbb{Z}^2\right)^* = \mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$. Its vertices are purple and its edges are drawn in green. The circuit surrounding $(0,0)$ is in bold.

$\longleftarrow$ An edge is present in the dual graph if and only if it does not cross an edge of the original graph.

Observe: $(0,0) \notin \infty$ cluster $\implies \exists$ circuit $\in \left(\mathbb{Z}^2\right)^*$ surrounding $(0,0)$ (and thus intersecting $\left(n+\frac{1}{2}, 0\right)$ for some $n \geq 0$).

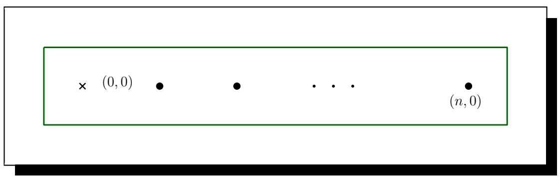


Figure 1.5: Smallest circuit surrounding $(0,0)$ and intersecting $\left(n+\frac{1}{2}, 0\right)$. Length : $2n+4$.

Thus,

$$\begin{aligned}
1-\theta(p) &\leq \sum_{n\geq 0} \mathbb{P}\left[\exists \text{ a circuit in the dual surrounding } (0,0) \text{ and crossing } \left(n+\frac{1}{2},0\right)\right] \\
&\leq \sum_{n\geq 0} \mathbb{P}\left[\exists \text{ a path of length } 2n+4 \text{ crossing } \left(n+\frac{1}{2},0\right)\right] \\
&\leq \sum_{n\geq 0} (4(1-p))^{2n+4} \qquad (\text{in the dual graph, } p_{edge}=1-p)
\end{aligned}$$

This quantity can be arbitrarily small if $p < 1$ is taken close from 1.

$\implies p_c < 1$

Theorem: (Aizenman, Kesten & Newman 1987, Burton-Keane 1989) Unicity of the giant component. If $p \in (0,1)$ is such that $\theta(p) > 0$,

$$\mathbb{P}_p\left[\exists! \infty \text{ cluster}\right] = 1.$$

Proof: Let the events

$$\begin{cases} C_{\leq 1} : \text{there is at most 1 } \infty \text{ cluster} \\ C_{<\infty} : \text{there is at most finitely many } \infty \text{ clusters} \\ C_{\infty} : \text{there are infinitely many } \infty \text{ clusters} \end{cases}$$

Observation: $\mathbb{P}_p$ is an ergodic measure of probability. So each event that is invariant by translation has probability 0 or 1red[1]. That is the case of the three previous events.
Proof of observation: Take any event A invariant under translation. Since the measure is finite and the σ-algebra is spanned by the 'cylinders' (*i.e.* the events depending on finitely many edges), for all $\varepsilon > 0$ there exists A_n depending on finitely many edges such that:

$$\mathbb{P}_p(A \Delta A_n) \leq \varepsilon.$$

For $x \in \mathbb{Z}^2$ sufficiently large, A_n and $\tau_x A_n$ (its translated by x) are independent:

$$\begin{aligned}
\mathbb{P}_p(A_n \cap \tau_x A_n) &= \mathbb{P}_p(A_n)\mathbb{P}_p(\tau_x A_n) \\
&= \mathbb{P}_p(A_n)^2.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{P}_p(A) = \mathbb{P}_p(A \cap A) &= \mathbb{P}(A \cap \tau_x A) \\
&\leq \mathbb{P}_p(A_n \cap \tau_x A_n) + 2\varepsilon \\
&\leq \mathbb{P}_p(A_n)^2 + 2\varepsilon \\
&\leq \mathbb{P}_p(A)^2 + 4\varepsilon \\
&\implies \mathbb{P}_p(A) \in \{0,1\}.
\end{aligned}$$

Proof of the theorem: Since $\theta(p) > 0$ there is an infinite cluster with non-null probability. Moreover, $C_{\leq 1} \subset C_{<\infty}$. There are only three possibilities:

$$\begin{pmatrix} \mathbb{P}_p(C_{\leq 1}) \\ \mathbb{P}_p(C_{<\infty}) \\ \mathbb{P}_p(C_{\infty}) \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

[1] Analogous to Kolmogorv's 0-1 law.

We have to rule out the first and third case. It is enough to show :

$$\mathbb{P}_p(C_{\leq 1}) = 1.$$

First step: Let us show : $\mathbb{P}_p(C_{<\infty} \setminus C_{\leq 1}) = 0$. (Eliminating the $(0, 1, 0)$-case.)

We only need to show: $\mathbb{P}_p(C_{<\infty}) > 0 \implies \mathbb{P}_p(C_{\leq 1}) > 0$. So we suppose that there is a *finite* number of infinite clusters. In this sub-universe, let us consider $\Lambda_n := [-n, n]^2$ the centered sub-grid of size $(2n + 1)^2$ and let $\mathcal{F}_n$ be the event:

$$\mathcal{F}_n := \{ \text{ all infinite clusters intersect } \Lambda_n \}.$$

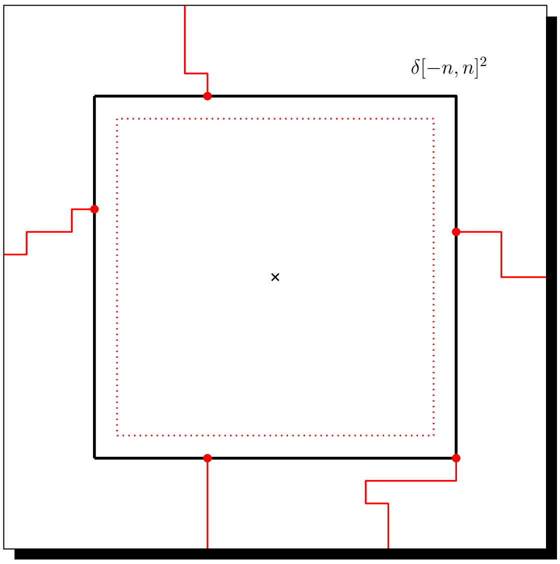


Figure 1.6: Λ_n and the infinite clusters (in red). There is a positive probability that all edges of the frontier (red-dotted) to be open.

$\mathcal{F}_n$ is independant of all edges inside Λ_n. Choose n_0 large enough so that with probability at least

$$\mathbb{P}(\mathcal{F}_{n_0}) \geq \frac{1}{2}\mathbb{P}(C_{<\infty}) > 0$$

all infinite clusters intersect the box Λ_{n_0}. Note that such n_0 must exist (this n_0 depends on the number of clusters, but such n_0 is found *a priori*, before exploring clusters). One may think about it in this way: there is only finitely many clusters, that have to be somewhere, so if there was no such n_0, we get a contradiction.

There is a positive probability that all edges of the boundary of Λ_{n_0} are open (red-dotted on the Figure **??**): $\mathbb{P}(\mathcal{F}_{n_0} \cap \text{ all edges in } \Lambda_{n_0} \text{ present }) > 0$. In this case, there is only one infinite cluster. Also

$$\mathbb{P}(C_{\leq 1}) > 0.$$

Second step: Suppose $\mathbb{P}(C_\infty) > 0$ (this is the $(0, 0, 1)$-case).
First, let us define the concept of trifurcation. Suppose that the origin $(0, 0)$ is in an infinite cluster. Remove the origin (and the adjacent bonds). In the new graph $\mathbb{Z}^d \setminus \{(0, 0)\}$, the previous cluster can either: 1°) Remain an infinite connected component (deprived of the origin and neighbour edges). 2°) Split into 2 infinite clusters (and perhaps with other finite clusters). 3°) Split into 3 (or maybe 4) infinite clusters (and perhaps with another finite cluster). We say that $(0, 0)$ is a **trifurcation** point in this latter case.

Define: Let $T_{(0,0)}$ be the event $\{(0, 0) \text{ is a trifurcation point }\}$. $x \in \mathbb{Z}^2$ is a trifurcation point if:

$$T_x := \tau_x T_{(0,0)} \text{ occurs.}$$

For $K \geq 3$, find n_0 large enough so that

$$\mathbb{P}\left(\text{(at least) } K \text{ infinite clusters intersect } \Lambda_{n_0} \right) \geq \frac{1}{2}\mathbb{P}\left(C_\infty\right).$$

$K := 3$ works for the demonstration. Suppose for now $K = 3$. These 3 infinite clusters intersect $\delta\Lambda_{n_0}$ at three points: A, B and C (say at distance at least three from each other, to ensure rewiring). Then, modify the construction so that only red edges appear. See Figure **??**: we merge the three infinite clusters on $(0,0)$. In this configuration, the origin becomes a trifurcation point. This means that:

$$\mathbb{P}(T_{(0,0)}) \geq \frac{1}{2}\mathbb{P}(C_\infty)(p(1-p))^{|Edges(\Lambda_{n_0})|}$$

(The probability of Λ_{n_0}'s internal configuration has been harshly majorized by $(p(1-p))^{|Edges(\Lambda_{n_0})|}$.)

Observe: If $(0,0)$ is a trifurcation point for some n, then it is also a trifurcation point for all $m \geq n$ (other edges could be present inside the boxes, note that above we just give a lower bound on the probability of a trifurcation point).

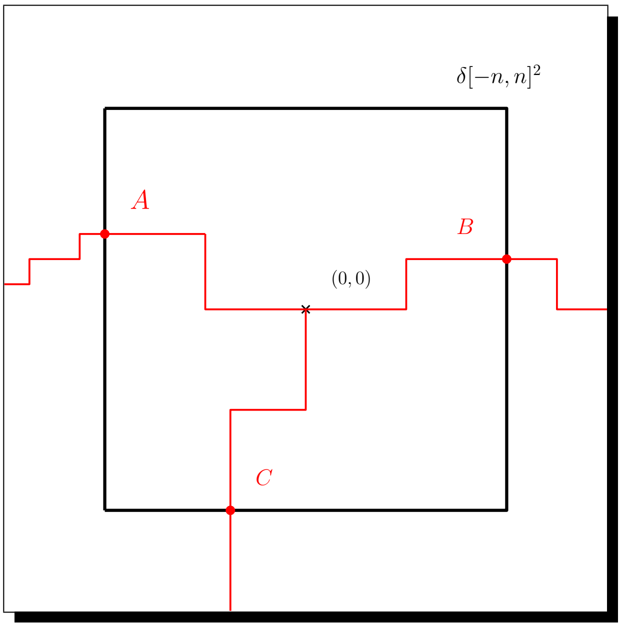


Figure 1.7: All edges in Λ_{n_0} except the red ones have been removed. $(0,0)$ thus becomes a trifurcation point.

Take a larger box of size $M \gg n_0$. We want to count the number $T := \sum_{x\in\Lambda_M} T_x$ of trifurcation points inside.

$$\mathbb{E}[T] = \mathbb{P}\left(T_{(0,0)}\right)|\Lambda_M|. \tag{1.1}$$

Do the following pealing inside Λ_M:

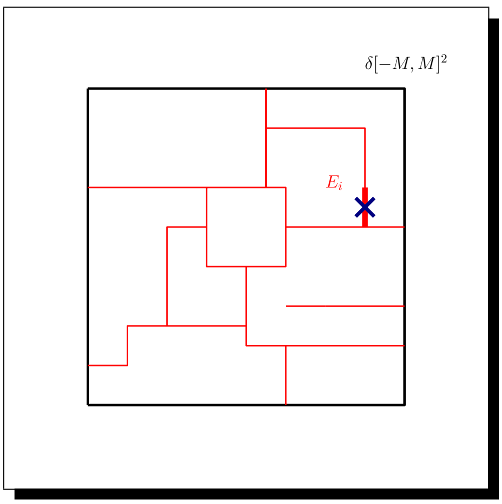


Figure 1.8: Remove the edges in Λ_M forming a cycle. The result is thus a forest.

Order edges inside Λ_M according to some (arbitrary) order $E_1, \dots, E_r$. If E_i forms a cycle, take it out.
$\implies$ New list of edges $F_1, \dots, F_{r'}$ without cycles.

If the new list $F \setminus \{F_i\}$ contains a cluster (can be a single vertex) non touching the boundary, remove all the vertices and edges of this cluster.
$\implies$ We end up with a **forest** (*i.e.* a set of disjoint trees) where **all leaves are on the boundary** $\partial\Lambda_M$.

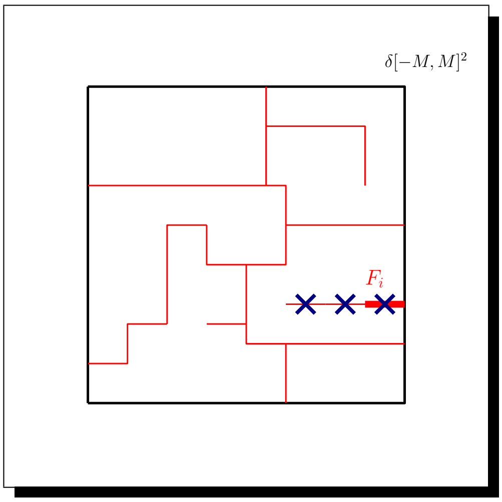


Figure 1.9: The removal of clusters of $F \setminus \{F_i\}$ without contact with the boundary.

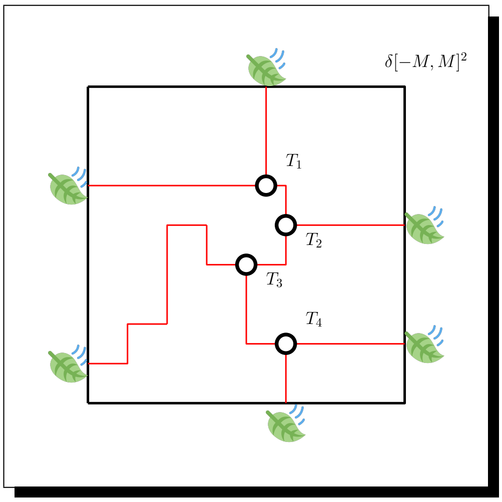


Figure 1.10: The trifurcation points T_x are the internal nodes of trees whose leaves are on the boundary of Λ_M.

Observe: The number of trifurcation points is less or equal to the number of leaves.
Proof of observation: Classical result of graph theory: the trifurcation points are the internal nodes of a tree. Let a tree of n nodes (m internal nodes of degree $\geq$ 3 and l leaves). Suppose first that there are no degree 2-nodes. Then we have: $n = m + l$. A tree of n nodes has exactly $n-1$ edges. Then, by double-counting the edges, we have:

$$2(n-1) = \sum_{v=1}^{n} deg(v) \geq l + 3m$$
$$2(l+m-1) \geq l + 3m$$
$$l - 2 \geq m,$$

and for trees without vertices of degree 2 the statement holds. On the other hand, as long there are degree 2 vertices y, iteratively contract the corresponding edges xy and yz to xz and observe that the contraction operation does not change the ratio betrween degree 1 vertices and vertices of degree ≥ 3. The observation follows.

In particular,

$$|T| \leq |\partial\Lambda_M|.$$

Coupled with (1.1), we obtain:

$$\mathbb{P}\left(T_{(0,0)}\right) = \frac{\mathbb{E}[T]}{|\Lambda_M|} \leq \frac{|\partial\Lambda_M|}{|\Lambda_M|} \to 0 \text{ when } M \to \infty.$$

Since we had

$$\mathbb{P}\left(T_{(0,0)}\right) \geq \frac{1}{2}\mathbb{P}(C_\infty)(p(1-p))^{|Edges(\Lambda_{n_0})|} \quad \text{where } n_0 \ll M \text{ was fixed,}$$

we must necessarily have

$$\mathbb{P}(C_\infty) = 0.$$

Lecture 2

Long-range percolation

Vertex set : $\mathbb{Z}$.

$$\forall i \neq j \in \mathbb{Z} = \begin{cases} \{i,j\} \text{ is an edge } (\textbf{open}) \text{ with probability} & p_{i,j}(\beta,\lambda) \\ \{i,j\} \text{ is not an edge } (\textbf{closed}) \text{ with probability} & 1-p_{i,j}(\beta,\lambda) \end{cases}$$

These events are all mutually independent. Moreother, for $\beta, \lambda > 0$, two intensity parameters, we define

$$p_{i,j}(\beta,\lambda) = \begin{cases} 1-e^{-\lambda} & \text{if } |i-j| = 1 \\ 1-e^{-\frac{\beta}{|i-j|^2}} & \text{if } |i-j| \geq 2 \end{cases}$$

Large edges are permitted, but less likely:
λ large $\Longrightarrow$ $p_{i,i+1}$ is likely.
β large $\Longrightarrow$ prob. of long edges large.
For fixed β, the larger the distance, the smaller the probability of an edge.

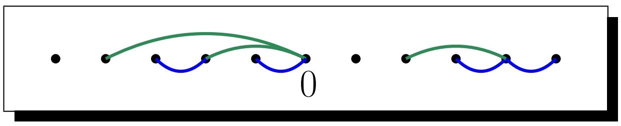


Figure 2.1: Long (depending on β) and short edges (depending on λ) drawn on $\mathbb{Z}$.

Same question: Let

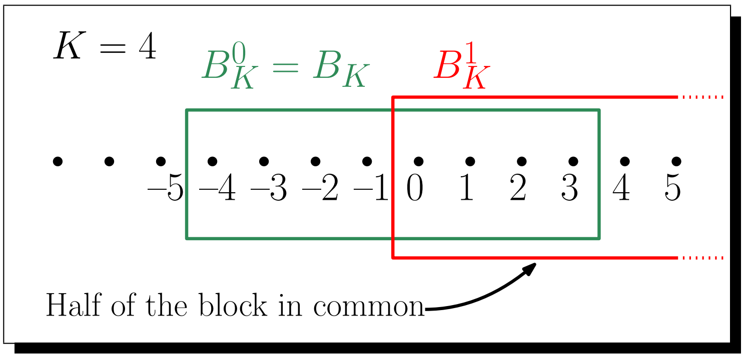
$$\theta(\beta,\lambda) = \mathbb{P}\,(0 \text{ is an } \infty \text{ cluster}).$$

Theorem: For every $\beta > 1$, there exists a large enough $\lambda < \infty$ such that

$$\theta(\beta,\lambda) > 0.$$

(In fact, if $\beta \leq 1$ then $\theta(\beta,\lambda) = 0$.)
Main object: $K \in \mathbb{Z}$, a K-Block centered at $i \in \mathbb{Z}$ is the set of integers

$$B_K^i = [K(i-1), \ldots, K(i+1)\,).$$



A block of size K has $2K$ elements. B_K^i and B_K^{i+1} share K elements.

Figure 2.2: Representation of $B_K^i = [K(i-1), \dots, K(i+1))$.

For $S \subseteq \mathbb{Z}$, a **cluster** in S is a connected component using only vertices and edges in S.
Define: For $\theta < 1$, K-Block B_K^i is θ-good if it contains a cluster of size at least $2\theta K$ elements. Otherwise, B_K^i is θ-bad.

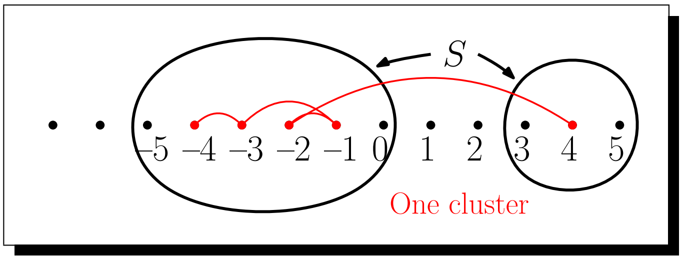


Figure 2.3: A cluster (in red) of five points and four edges.

Let

$$p(K, \theta) = \mathbb{P}(B_K \text{ is } \theta\text{-bad }).$$

Proof idea: Clusters of size K and density θ will merge to create larger clusters at scale CK with density $\theta' = \theta - \frac{C_0}{C}$. (We may loose a few clusters at size K since θ' is a bit smaller.)

Key lemma: Let $\beta > 1$, $\frac{3}{4} < \theta_\infty < 1$ such that $\theta_\infty^2 \beta > 1$, then there exists C_0 large enough such that for every $\lambda > 0$, $\theta \geq \theta_\infty$ and every $C \geq C_0$, $K \geq 2$

$$p(CK, \theta - \frac{C_0}{C}) \leq \frac{1}{100} p(K, \theta) + 2C^2 p(K, \theta)^2.$$

Question: Why $\theta_\infty > \frac{3}{4}$?

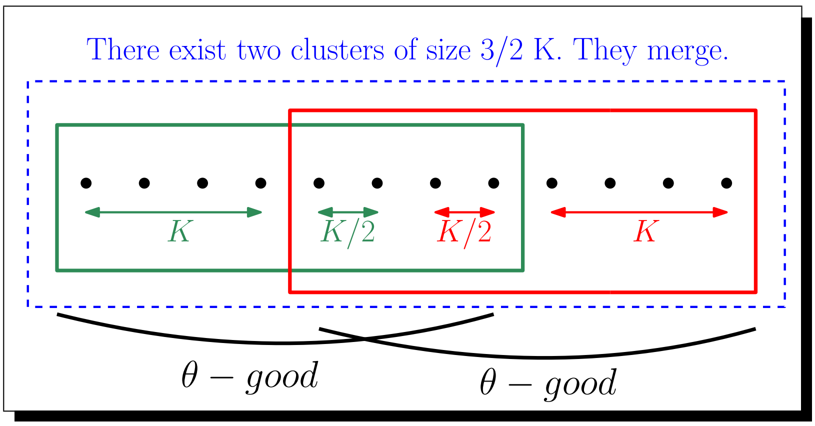


Figure 2.4: The two clusters θ-good (with $\theta > \frac{3}{4}$) straddle each other, merge and create a cluster which makes the block B_{2K} also θ-good.

Proof of Lemma : Let define $C(B^i_K)$ the largest cluster in B^i_K. For $|i| \leq C$, let E_i be the event

$$B^i_K \text{ is } \theta\text{-bad but } B^j_K \text{ is } \theta\text{-good for } j \in \{-(C-1), \ldots, C-1\} \setminus \{i-1, i, i+1\}.$$

Define: $F_i = E_i \cap \{B_{CK} \text{ is } \theta'\text{-bad}\}$ for $\theta' := \theta - \frac{C_0}{C}$.

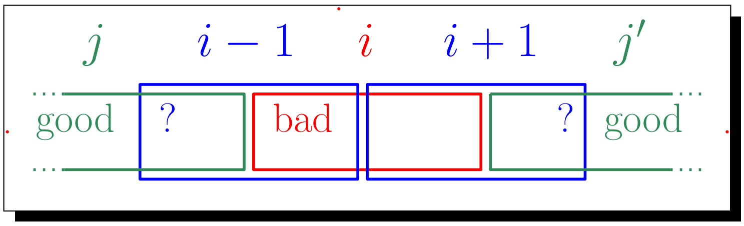


Figure 2.5: The event E_i.

If B_{CK} is θ'-bad $\implies$ largest cluster in the block is of size $\leq 2C\theta' K = 2C\theta K - 2C_0 K$. Therefore,

1. Either F_i occurs for some $|i| \leq C$.
2. Two disjoint K-Blocks must be θ-bad.

By a union bound,

$$p(CK, \theta') \leq \underbrace{\sum_{i=-(C-1)}^{C-1} \mathbb{P}[F_i]}_{1.} + \overbrace{\mathbb{P}(\exists \text{ two disjoint } \theta\text{-bad } K\text{-Blocks})}^{2.}.$$

2. $\mathbb{P}(\exists \text{ two disjoint } \theta\text{-bad } K\text{-Blocks}) \leq \binom{2(C-1)}{2} p(K, \theta)^2.$

1. If all boxes B^j_K with $|j| \leq C - C_0$ are θ-good, then B_{CK} is θ'-good. We may also assume $|i| \leq C - C_0$.

Let $|i| \leq C - C_0$,

$$\begin{aligned} \mathbb{P}(F_i) &= \mathbb{P}(E_i)\mathbb{P}(B_{CK} \text{ is } \theta'\text{-bad} \mid E_i) \\ &\leq p(K, \theta)\mathbb{P}(B_{CK} \text{ is } \theta'\text{-bad} \mid E_i). \end{aligned}$$

Define:
$C^- =$ Union of all clusters $C(B^j_K)$ for $j \leq i-2$.
$C^+ =$ Union of all clusters $C(B^j_K)$ for $j \geq i+2$.

The definition of C^-, C^+ do not tell us anything about edges from C^- to C^+. If largest clusters in C^- and C^+ merge, then B_{CK} is θ'-good.

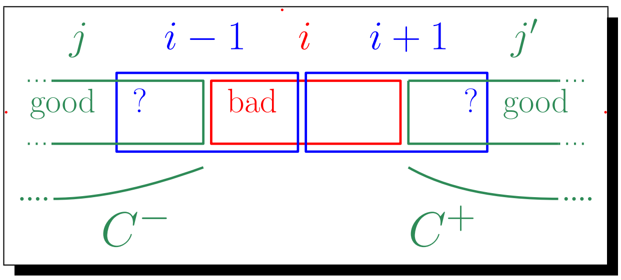


Figure 2.6: Definition of C^+ and C^-.

Observe:

$$|C^+ \cap \{iK + K, \ldots, iK + K + 2lK\}| \geq 2\theta K l \qquad \text{for each } l \geq 1 \text{ by } \theta\text{-goodness.}$$

Let $iK < y_1 < y_2 < \ldots$ are all elements of C^+. For $l = \lceil \frac{b}{2\theta K} \rceil$,

$$\begin{aligned} y_b &\leq iK + K + 2K \left\lceil \frac{b}{2\theta K} \right\rceil \\ &\leq iK + \frac{3K + b}{\theta}. \end{aligned}$$

Similarly, let $iK > x_1 > x_2 > \ldots$ all elements of C^-.

$$x_a \geq iK - \frac{3K + a}{\theta}.$$

$$\implies \qquad \theta \left(y_b - x_a \right) \leq 6K + a + b.$$

Check all edges:

$$\begin{aligned} \mathbb{P} \left(\text{ no edge of largest clusters of } C^- \text{ to } C^+ \right) &= \prod_{x \in C^-} \prod_{y \in C^+} \mathbb{P} \left(\{x, y\} \text{ is not an edge} \right) \\ &\leq e^{-\beta\theta^2 \sum_{a=1}^{|C^-|} \sum_{b=1}^{|C^+|} \frac{1}{\theta^2 (y_b - x_a)^2}}. \end{aligned}$$

Claim without proof: The Sum-Integral comparison gives the bound:

$$\sum_{a=1}^{|C^-|} \sum_{b=1}^{|C^+|} \frac{1}{\theta^2 (y_b - x_a)^2} \geq \log \left(\frac{C - |i|}{8} \right).$$

Thus

$$\mathbb{P} \left(\text{ no edge of largest clusters of } C^- \text{ to } C^+ \right) \leq \left(\frac{C - |i|}{8} \right)^{\beta\theta^2}$$

Then, for C_0 large enough

$$\sum_{-(C-1)}^{C-1} \mathbb{P}(F_i) \leq p(K, \theta) \sum_{-(C-C_0)}^{C-C_0} \left(\frac{8}{C - |i|} \right)^{\beta\theta^2} \leq \frac{1}{100} p(K, \theta).$$

Proof of the theorem:

Let $\beta > 1$, $\theta_\infty \in (\frac{3}{4}; 1)$, $\beta\theta_\infty^2 < 1$.
Let $\theta_p < 1$ and $C_p \geq C_0$ and set

$$\begin{aligned} C_{n+1} &= (n+1)^3 C_1 \quad \text{and} \\ \theta_{n+1} &= \theta_n - \frac{C_0}{C_{n+1}} \quad (\text{s.t. } \forall n, \theta_n > \theta_\infty > \frac{3}{4}.) \end{aligned}$$

Initially, start with blocks of size $K_1 = C_1$ and λ so large that

$$p(C_1, \theta_1) \leq C_1 e^{-\lambda} \leq \frac{1}{1000 C_1^2}.$$

Set up the blocks

$$K_1 = C_1,$$
$$K_{n+1} = C_{n+1} K_n.$$

Set $u_n = p(K_n, \theta_n)$. By the "Key Lemma",

$$u_{n+1} \leq \frac{1}{100} u_n + 2 C_{n+1}^2 u_n^2.$$

By induction,

$$\forall n,\ C_n^2 u_n \leq \frac{1}{1000}$$
$$\implies \quad \mathbb{P}\left(B_{K_n} \text{ is } \theta\text{-good}\right) \geq 1 - \frac{1}{1000 C_n^2} \geq \frac{1}{2}.$$
$$\implies \quad \mathbb{P}\left(0 \in \text{ infinite component}\right) > 0.$$

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