Mathematics of X-ray Computed Tomography: Just Enough Physics and Continuous Theory

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Università degli Studi di Ferrara
- Ex labore fructus -

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European Women in Mathematics

(Rough) Outline

Lecture 1 – Monday 5th January

- What is tomography? Just enough physics
- A splash of theory on the Radon transform and Filtered Backprojection

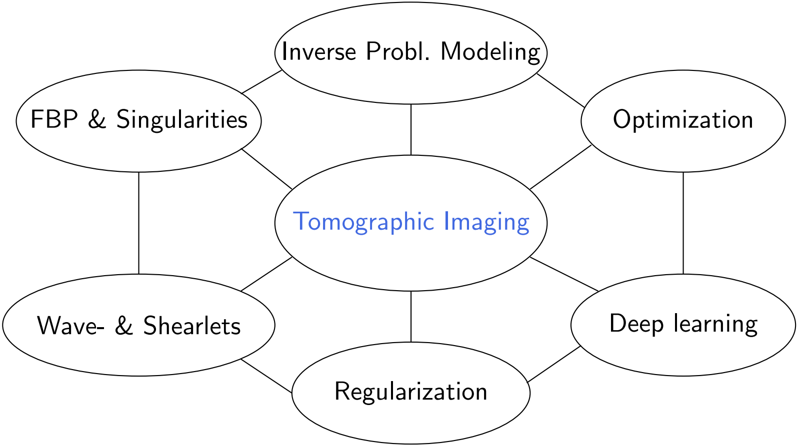
Lecture 2 – Tuesday 6th January

- Regularisation methods to solve (tomographic) inverse problems
- A very fast wavelet tour (of signal processing)

Lecture 3 – Wednesday 7th January

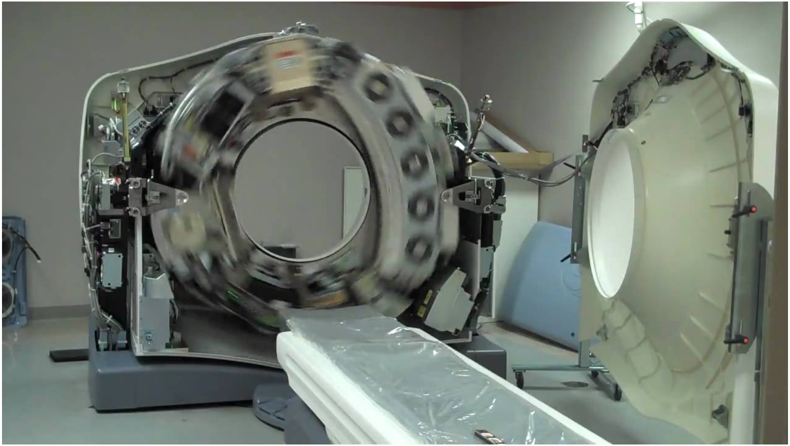
- Nods to convex optimization
- Short introduction to learned reconstruction methods

(Rough) Outline But With a Drawing

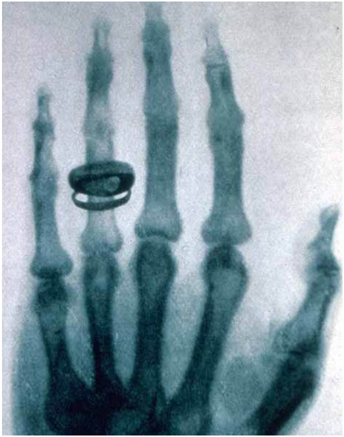


Modern CT Scanners

Modern CT Scanners: Inside



The Story Begins With Röntgen's Discovery of X-rays

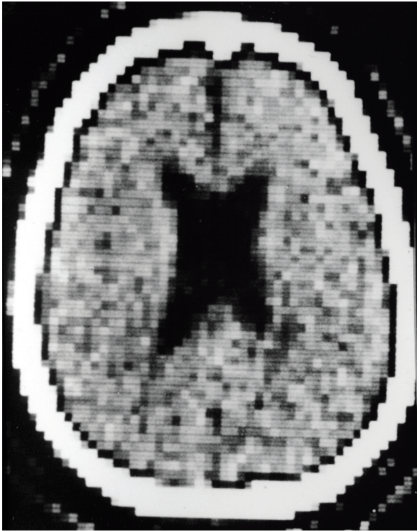


1895: Wilhelm Conrad Röntgen discovers X-rays

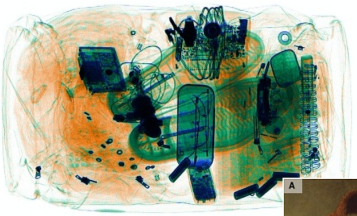
1901: Röntgen is awarded the Nobel Prize in Physics

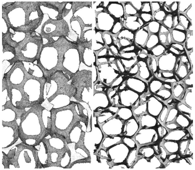
Several Decades Later . . .

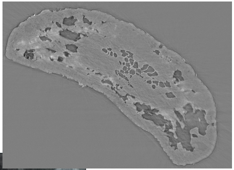
1979: Godfrey Hounsfield (top) and Allan McLeod Cormack receive the Nobel prize for developing X-ray tomography.

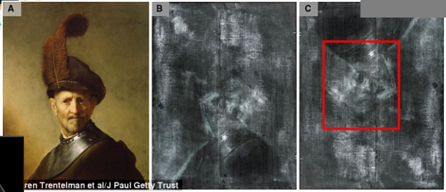


Nowadays: Deluge of Applications





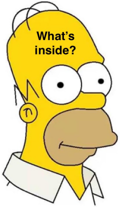




What is Computed Tomography?

Tomography: derives from *tomos* (a section or slice) and *graphos* (to describe)

CT is a non-invasive device that provides information about the inside of an object by taking measurements from the outside (indirect information).



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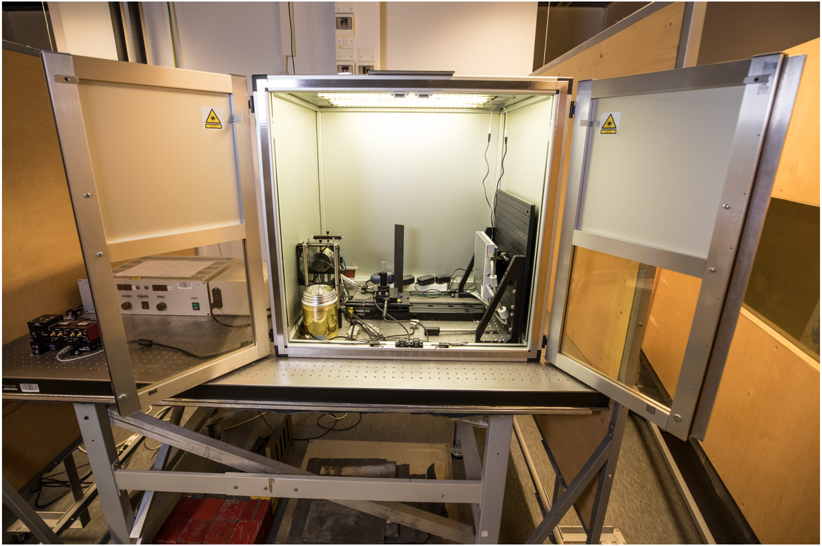
At the core:

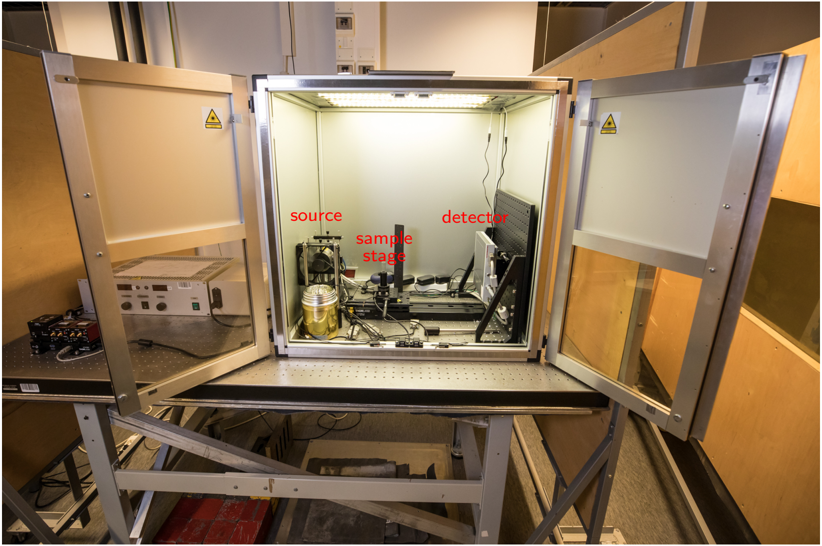
- Measurements are taken exploiting the transmission of waves or particles (e.g., X-rays)
- The intensity of particles transmission is attenuated by the material through which they travel

In Practice: Experimental Imaging Setup



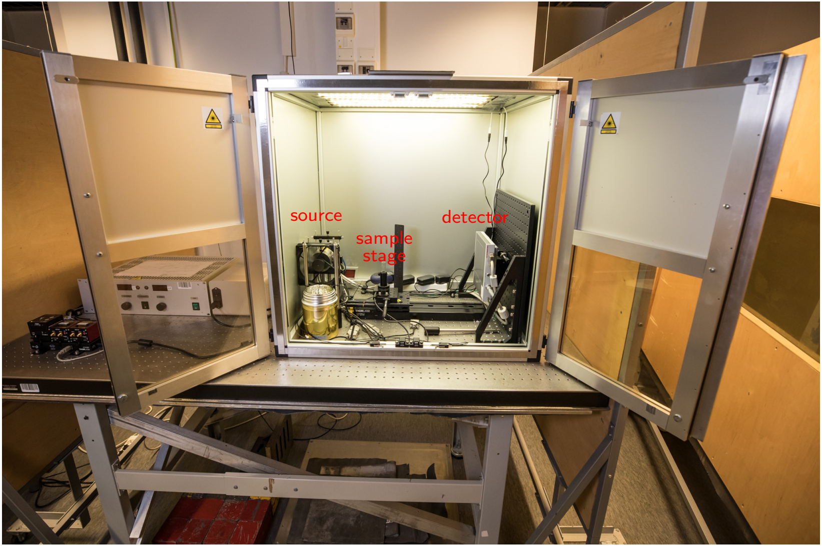
μCT system at University of Helsinki

In Practice: Experimental Imaging Setup



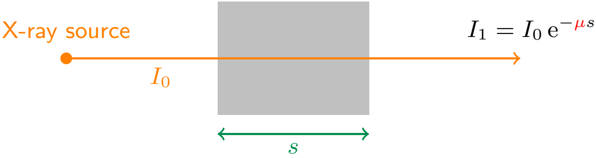
Primary components of μCT system: source, target, detector

In Practice: Experimental Imaging Setup



Source emits X-rays → passing through the target → measured by detector

Toy Example: A Line Inside Homogeneous Matter

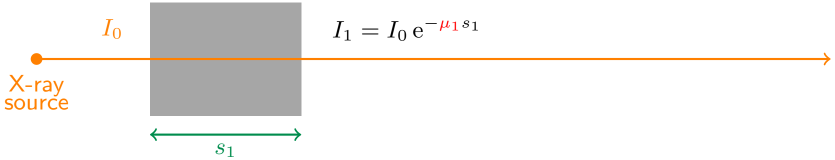


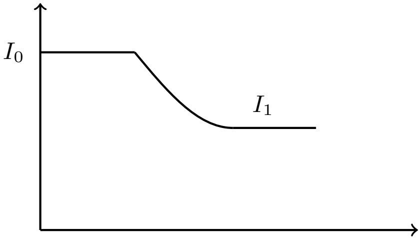
I_0: initial intensity of the X-ray

s: length of the path of the X-ray inside the object (particles are assumed to more or less travel in straight lines)

$\mu > 0$: X-ray attenuation coefficient

Toy Example: Two Homogeneous Blocks





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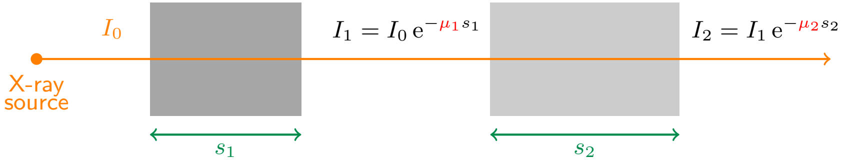
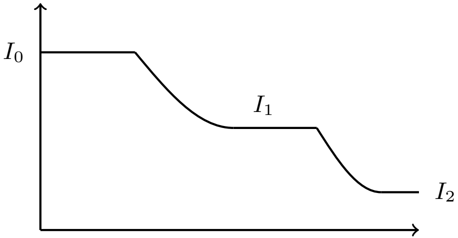
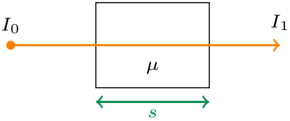




Absorption in the Target: the Beer-Lambert Law

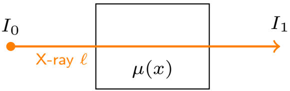
Homogeneous material:

$$I_1 = I_0 \; e^{-\mu s}$$



Non-homogeneous material:

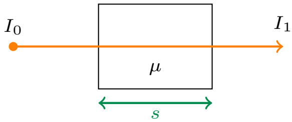
$$I_1 = I_0 \; e^{-\int_\ell \mu(x) \, dx}$$



Absorption in the Target: Energy Dependence

Homogeneous material:

$$I_1 = I_0 \, e^{-\mu s}$$



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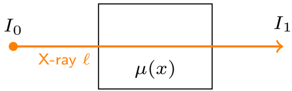


In reality, to accurately describe the physical process an **energy-dependent non-linear integral** model would be necessary:

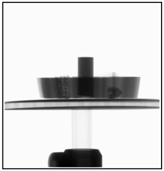
$$I_1 = \int I_0(E)\, e^{-\int_\ell \mu(E,x)\, dx} dE$$

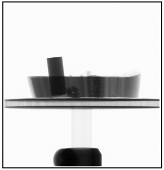
Usually, this energy-dependence is neglected and an effective absorption coefficient $\mu_{\text{eff}}(x)$ is assumed.

Imaging at the Detector

The detector measures a resulting X-ray projection image:

- The most common energy integrating detectors (EIDs) provide a monochromatic image





Imaging at the Detector

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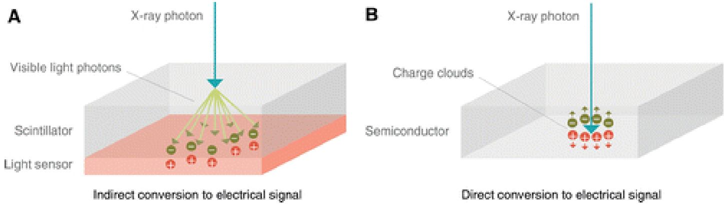
- Photon counting detectors (PCDs) can detect photons of different energies and allow for multi-energy X-ray (nonlinear)



[Image credits: Willemink et al., Radiology, 2018]

Transforming the Measurement for the Inverse Problem

The Beer-Lambert law connects the initial and final intensities of an X-ray:

$$I_1 = I_0 \, e^{-\int_\ell \mu(x)\, dx} \qquad \Longleftrightarrow \qquad -\log\left(\frac{I_1}{I_0}\right) = \int_\ell \mu(x)\, dx$$

where $-\log(I_1/I_0)$ models the total attenuated energy according the attenuation along the path ℓ.

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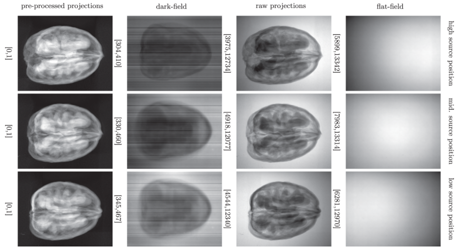
Before obtaining processed measurements, need to compensate "detector noise":

- **Dark-field** recorded with source off: detector offset count
- **Flat-field** with source on: the beam profile



[Der Sarkissian et al., Scientific Data, 2019]

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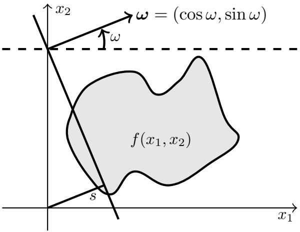
As a result, during a tomographic scan:

- I_0 is known from calibration and I_1 from measurements
- I_1 is measured along many lines $\ell_{(\boldsymbol{\omega},s)}$ to get many line integral values through the object
- The intensity I_1 is called the *transmission*, while the corresponding $-\log(I_1/I_0)$ is called absorption or **projection**, and a collection of projections is called a **sinogram**

Beer-Lambert Law and Radon Transform

The problem of recovering the attenuation function (linearised measurement) can be mathematically modelled by the **Radon transform**, which can be understood as an integration of the function $f : \mathbb{R}^2 \to \mathbb{R}_+$ over lines.

Through the identifications $f(\boldsymbol{x}) = \mu(\boldsymbol{x})$ and $\mathcal{R}(f) = -\log(I_1/I_0)$, the Beer-Lambert law is connected to the **Radon transform**:

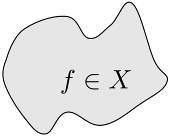


$$\mathcal{R}(f)(\boldsymbol{\omega}, s) = \int_{-\infty}^{\infty} f(s\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp})\, d\tau$$
$$= \int_{\ell(\boldsymbol{\omega}, s)} f(\boldsymbol{x})\, d\boldsymbol{x},$$

where $\ell = \ell(\boldsymbol{\omega}, s) = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} = s\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp}, \tau \in \mathbb{R}\}$ with $\boldsymbol{\omega} = (\cos(\omega), \sin(\omega))$ and $(\boldsymbol{\omega}, s) \in S^1 \times \mathbb{R}$.

In Practice: How to Formulate CT as a Mathematical Problem

- We aim at imaging a target (e.g., human chest) $f \in X = L^2(\Omega)$ with $f : \mathbb{R}^d \to \mathbb{R}_+$ in a bounded domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$.



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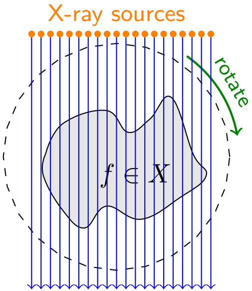
- The process of emitting X-rays that travel through target $f \in X$ is called the **forward problem/model** ($\rightsquigarrow$ Radon transform).



Forward Problem

$\mathcal{R}(f)$

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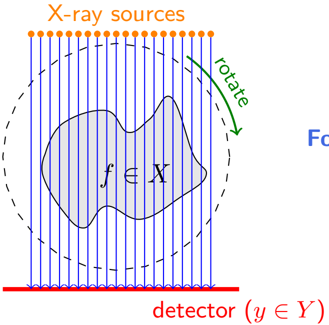

Forward Problem

$\mathcal{R}(f) = y$

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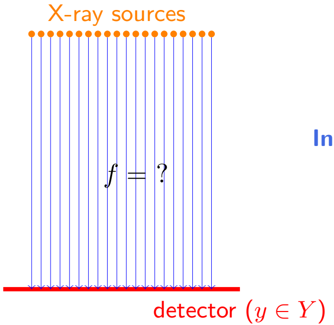
- Reconstructing f from the measured data y is then consequently the **inverse problem**.



Inverse Problem

$f = ?$

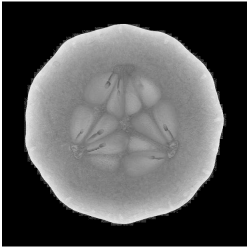
The Basic Linear Inverse Problem

The forward model can be stated as:

$$Af = y^\delta$$

where:

$A : X \to Y$	The linear forward operator (defining the scanning geometry)
$f \in X$	The unknown/quantity of interest (linearised attenuation coefficient)



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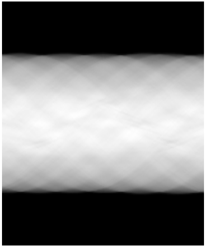

measurement data

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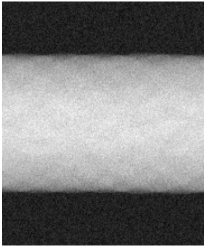
$\epsilon \in Y$	Measurement noise (with $\|\epsilon\|_Y \leq \delta$)



noisy measurement data

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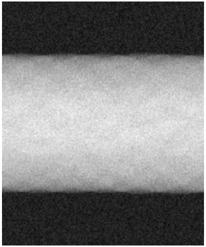

noisy measurement data

Inverse problem

Given noisy measurements y^δ, determine f.

Ill-Posedness and Hadamard's Conditions

The problem

Given y^δ, determine f with $Af = y^\delta$

is well-posed if the following conditions hold true [Hadamard, 1903]:

1. A solution exists (surjectivity)
2. The solution is unique (injectivity)
3. The solution depends continuously on the input (stability)

If one of these conditions fails, the problem is said **ill-posed**.

A splash of continuous theory: Radon transform

Recall the the Radon transform

$$y(\boldsymbol{\omega}, s) = \mathcal{R}(f)(\boldsymbol{\omega}, s) = \int_{\ell(\boldsymbol{\omega},s)} f(\boldsymbol{x})\ d\boldsymbol{x}$$

- It is dependent on an **angle** on the unit circle $\omega \in (0, 2\pi]$, with $\boldsymbol{\omega} = (\cos(\omega), \sin(\omega))$, and a **signed distance** $s \in \mathbb{R}$.
- $d\boldsymbol{x}$ denotes the one-dimensional (Lebesgue) measure along the line $\ell(\boldsymbol{\omega}, s) = \{\boldsymbol{x} \in \mathbb{R}^2\ :\ \boldsymbol{x} \cdot \boldsymbol{\omega} = s\}$, i.e., we only integrate over single lines for each ω and s!
- The measurement $y = \mathcal{R}(f)$ is a function defined on the parametrization of the infinite unit cylinder in $\mathbb{R}^3$:

$$C^2 = \{(\boldsymbol{\omega}, s)\ :\ \omega \in [0, 2\pi),\ s \in \mathbb{R}\}.$$

- Notice that $\mathcal{R}(f)(\boldsymbol{\omega}, s) = \mathcal{R}(f)(-\boldsymbol{\omega}, -s)$, so we can take $\omega \in (0, \pi]$.

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Integrating over lines for each angle $\boldsymbol{\omega} = (\cos(\omega), \sin(\omega))$ for $\omega \in (0, \pi]$ results in the sinogram.

Boundedness of the Radon Transform

Given a linear operator $\mathcal{A} : X \to Y$ between two Banach/Hilbert spaces, we say that the operator is **bounded**, if there exists a constant $C > 0$ such that

$$\|\mathcal{A}f\|_Y \leq C\|f\|_X, \qquad \text{for all} \;\; f \in X.$$

The smallest such C is the operator norm $\|\mathcal{A}\|_{\mathsf{op}} = \|A\|_{X\to Y} = C$. In particular, a **bounded linear operator is continuous**.

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Theorem

Let $\mathcal{R}$ be the Radon transform, $\Omega_1 \subset \mathbb{R}^2$ a bounded set and $\operatorname{supp}(f) \subset \Omega_1$, so that the integration reduces to $\ell(\boldsymbol{\omega}, s) \cap \Omega_1$. Then $\mathcal{R}$ is a **bounded linear operator** from $L^2(\Omega_1)$ to $L^2(C^2)$. That is, $\exists\ c > 0$ such that:

$$\|\mathcal{R}f\|_{L^2(C^2)} \leq c\|f\|_{L^2(\Omega_1)}, \qquad \forall\ f \in L^2(\mathbb{R}^2).$$

The Adjoint Operator or Backprojection

The Radon transform defines the forward operator $X \to Y$ (image to measurement), **for reconstruction we also need a mapping from $Y \to X$ (measurement to image)**.

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For that purpose, we will first use the **adjoint operator** $\mathcal{R}^*$, which is defined through the inner product by the relationship:

$$\langle \mathcal{R}g, h \rangle_Y = \langle g, \mathcal{R}^* h \rangle_X, \qquad \text{for all } \; g \in X, \; h \in Y,$$

where the inner product is the inner product in $L^2(\Omega)$ (where our image f is defined), with $\Omega \subset \mathbb{R}^2$:

$$\langle g, h \rangle_{L^2(\Omega)} = \int_\Omega g(z)h(z)dz.$$

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With this we can define the adjoint operator for the Radon transform, commonly refereed to as **backprojection**.

The Adjoint Operator or Backprojection

For the functions $f(\boldsymbol{x})$, with $\boldsymbol{x} \in \Omega \subset \mathbb{R}^2$, and $y(\boldsymbol{\omega}, s)$ with $(\boldsymbol{\omega}, s) \in S^1 \times \mathbb{R}$, the adjoint $\mathcal{R}^*$ (called **backprojection**) of the Radon transform is given by:

$$(\mathcal{R}^* y)(\boldsymbol{x}) = \int_0^\pi y(\boldsymbol{\omega}, \boldsymbol{x} \cdot \boldsymbol{\omega})\ d\omega$$

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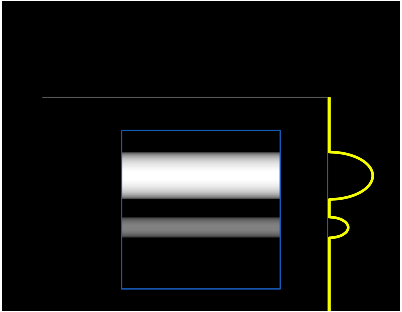

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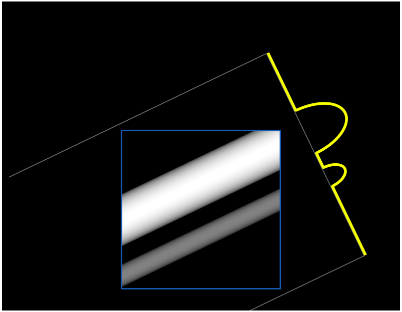

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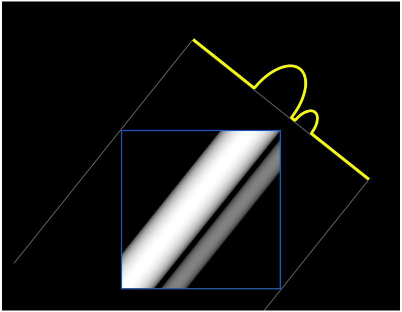

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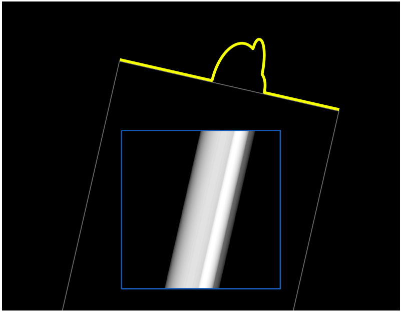

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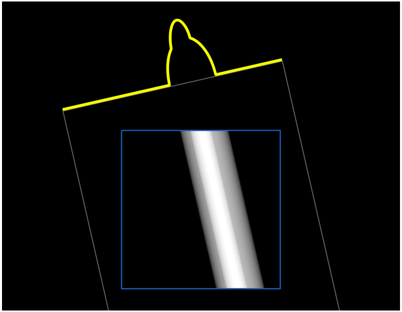

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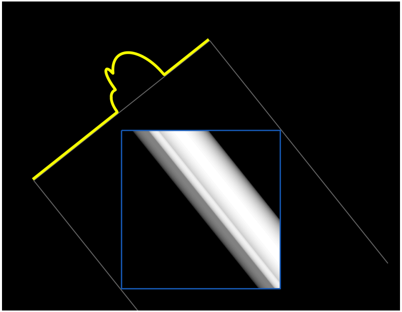

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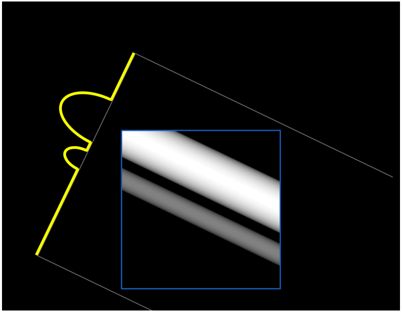

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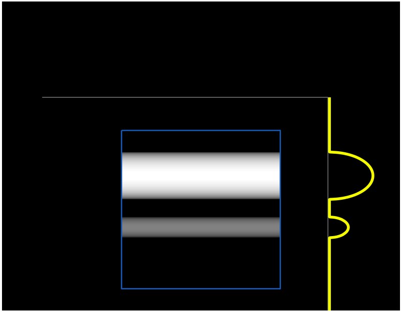

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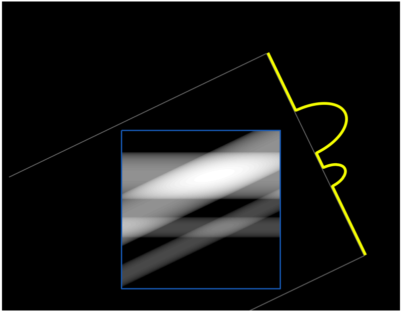

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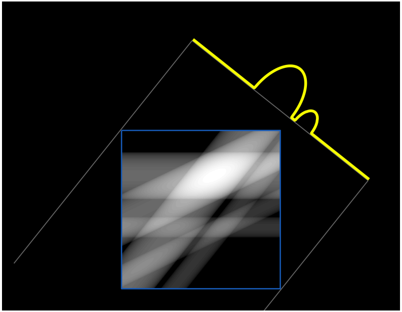

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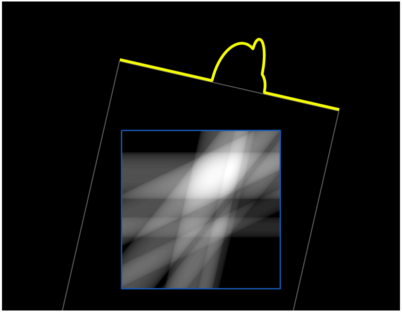

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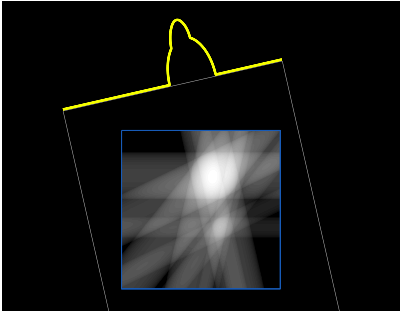

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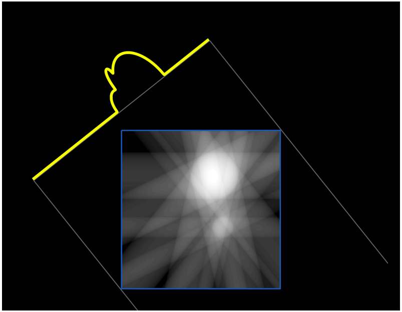

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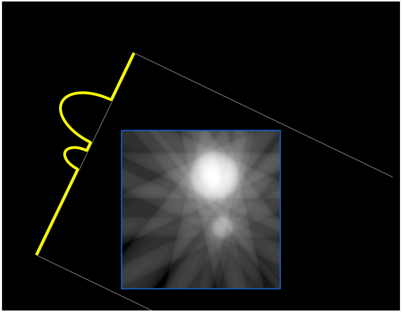

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Regularity of the Radon Transform

Theorem [2.10 with $n = 2,\ \alpha = 0$, Natterer & Wübbeling 2001]

Let $f \in L^2(\Omega_1)$, then there exist two constants $c,\ c' > 0$ such that

$$c\|f\|_{L^2(\Omega_1)} \leq \|\mathcal{R}f\|_{H^{1/2}(C^2)} \leq c'\|f\|_{L^2(\Omega_1)}.$$

Roughly speaking, the Radon transform is a continuous linear operator from $L^2(\Omega_1)$ to $H^{1/2}(C^2)$ and hence $\mathcal{R}f$ is smoother than f by half a derivative.

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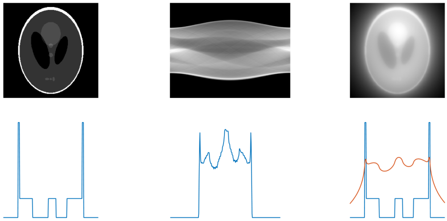
A similar result holds for the backprojection, which means that the projection of f followed by a backprojection, i.e. $\mathcal{R}^*\mathcal{R}f$, smooths f by a full derivative.



f $\qquad$ $\mathcal{R}f$ $\qquad$ $\mathcal{R}^*\mathcal{R}f$

Ill-Posedness of the Radon Transform

✓ **Injectivity.** We have the following result:

Theorem

Let $S_0 \subset S^1$ be a set of **infinite** many directions and let $f \in L^2(\Omega_1)$. If $(\mathcal{R}f)(\boldsymbol{\omega}, \cdot) = 0$ for every $\omega \in S_0$, then $f = 0$.

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However:

Caveat!

This is **not** true for **finitely** many directions!! Which means, this does not hold in the discrete case!

Ill-Posedness of the Radon Transform

✓ **Injectivity.** The infinite dimensional Radon transform is injective.

✗ **Surjectivity.** The range of $\mathcal{R}$ is an **infinite** dimensional proper subspace (i.e., $H^{1/2}(C^2)$) of $L^2(C^2)$ (see again Theorem 2.10 in [Natterer & Wübbeling 2001]).

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Moreover:

Corollary

The Radon transform is a **compact operator** with infinite-dimensional range and hence it has an open range in $L^2(C^2)$.

✗ **Stability.** The range of the Radon transform is open so there is **no stability.**

Ill-Posedness of the Radon Transform

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Can We Invert the Radon Transform?

Johann Radon (1887-1956)

How to reconstruct a function from its line integrals:

$$f(P) = -\frac{1}{\pi}\int_0^\infty \frac{d\overline{F_p}(q)}{q}$$

SITZUNG VOM 30. APRIL 1917.

Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten.

Von

JOHANN RADON.

Satz III: Der Wert von f ist durch F eindeutig bestimmt und läßt sich folgendermaßen berechnen:

(III) $$f(P) = -\frac{1}{\pi}\int_0^\infty \frac{d\overline{F}_P(q)}{q}.$$

[from Johann Radon's 1917 Seminal Paper]

Radon's Inversion Formula: Modern Version

Let $f : \Omega \to \mathbb{R}$ and $y = \mathcal{R}f$ then, for $\boldsymbol{x} \in \Omega \subset \mathbb{R}^2$, f can be obtained as:

$$f(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_0^{\pi} \int_{\mathbb{R}} \frac{\partial_s y(\boldsymbol{\omega}, s)}{\boldsymbol{x} \cdot \boldsymbol{\omega} - s} \, ds \, d\omega$$

with $\boldsymbol{\omega} = (\cos(\omega), \sin(\omega))$.

Note: Practical reconstructions are **not** directly based on this formulation. However, similar ingredients are found:

- Integration over s with a function $(\boldsymbol{x} \cdot \boldsymbol{\omega} - s)^{-1}$ $\rightsquigarrow$ **Convolution**
- Integration over ω $\rightsquigarrow$ **Backprojection**

Convolution Theorem for the Radon Transform

Recall that given two functions $h,\ k : \mathbb{R}^n \to \mathbb{R}$:

$$(h * k)(\boldsymbol{u}) = \int_{\mathbb{R}^n} h(\boldsymbol{z})k(\boldsymbol{u} - \boldsymbol{z})\ d\boldsymbol{z} = \int_{\mathbb{R}^n} h(\boldsymbol{u} - \boldsymbol{z})k(\boldsymbol{z})\ d\boldsymbol{z}$$

The convolution theorem establishes a connection to the Fourier transform:

$$\mathcal{F}(h * k)(\xi) \ = \ \hat{h}(\xi)\hat{k}(\xi)$$

Convolution in image space $=$ Multiplication in Fourier space

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In fact, we can use convolutions to establish a connection between backprojected data and image space.

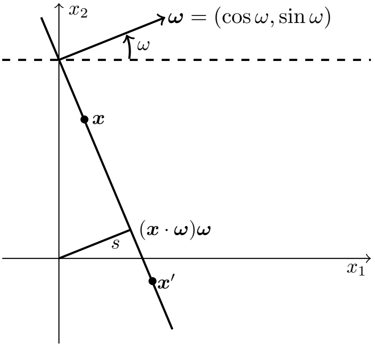
Radon Convolution Theorem

Let $v : C^2 \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$, then we have:

$$(\mathcal{R}^* v) * f = \mathcal{R}^*(v * \mathcal{R}f)$$

Conv. with BP kernel in image space $=$ BP of conv. in data space

Filtered Backprojection – Part I



First of all, notice that for each $\boldsymbol{z} \in \mathbb{R}^2$, we have:

$$\exists\ \tau \in \mathbb{R} \text{ s.t. } (\boldsymbol{x}\cdot\boldsymbol{\omega})\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp} = \boldsymbol{x}'$$
$$\Rightarrow\ \exists\ t \in \mathbb{R} \text{ s.t. } \boldsymbol{x} + t\boldsymbol{\omega}^{\perp} = \boldsymbol{x}'$$

Then, let's start by computing $\mathcal{R}^*\mathcal{R}f$:

$$\mathcal{R}^*\mathcal{R}f(\boldsymbol{x}) = \int_0^{\pi} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{x}\cdot\boldsymbol{\omega})\ d\omega = \int_0^{\pi}\int_{-\infty}^{\infty} f((\boldsymbol{x}\cdot\boldsymbol{\omega})\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp})\ d\tau d\omega$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x} + t\boldsymbol{\omega}^{\perp}) dt d\omega$$

Filtered Backprojection – Part II

Next, by using polar coordinates we get:

$$\begin{aligned}\mathcal{R}^*\mathcal{R}f(\boldsymbol{x}) &= \int_0^{\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x}+t\boldsymbol{\omega}^{\perp})dtd\omega \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{f(\boldsymbol{x}+t\boldsymbol{\omega}^{\perp})}{t}\ t\ dtd\omega = \int_{\mathbb{R}^2} \frac{f(\boldsymbol{x}+\boldsymbol{z})}{\|\boldsymbol{z}\|}\ d\boldsymbol{z}\end{aligned}$$

where $\boldsymbol{z} = (z_1, z_2)$ with $z_1 = -t\sin(\omega)$ and $z_2 = t\cos(\omega)$.

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Then, (with a trivial change of variables) we have:

$$\mathcal{R}^*\mathcal{R}f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \frac{f(\boldsymbol{x}+\boldsymbol{z})}{\|\boldsymbol{z}\|}\ d\boldsymbol{z} = \int_{\mathbb{R}^2} \frac{f(\boldsymbol{z})}{\|\boldsymbol{x}-\boldsymbol{z}\|}\ d\boldsymbol{z} = (f * v)(\boldsymbol{x})$$

where $v(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$ and $*$ denotes the convolution defined earlier.

Filtered Backprojection – Part III

Now, notice that $\widehat{v}(\xi) = \frac{1}{|\xi|}$, therefore by applying the Fourier transform we get:

$$\mathcal{F}(\mathcal{R}^*\mathcal{R}f)(\xi) = \mathcal{F}(f * v)(\xi) = \hat{f}(\xi)\hat{v}(\xi) = \frac{\hat{f}(\xi)}{|\xi|}$$

Finally, by multiplying both sides by $|\xi|$ and applying the inverse Fourier transform, we end up with:

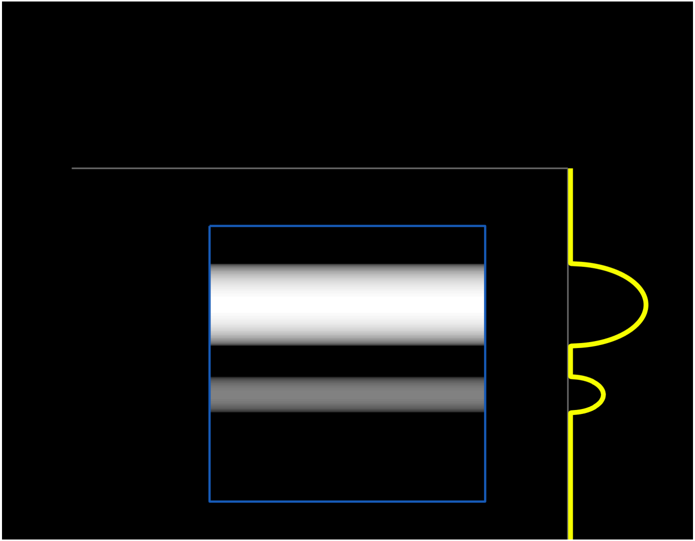
$$\Lambda\mathcal{R}^*\mathcal{R}f(\boldsymbol{x}) = \mathcal{F}^{-1}(|\xi|\mathcal{F}(\mathcal{R}^*\mathcal{R}f))(\boldsymbol{x}) = f(\boldsymbol{x}).$$

Hence, $\Lambda\mathcal{R}^*$ acts as a left inverse for the Radon transform $\mathcal{R}$, where

$$\Lambda\mathcal{R}^* = \mathcal{F}^{-1}|\xi|\mathcal{F}\mathcal{R}^*$$

is the Filtered Backprojection (FBP) operator. Notice how the FBP operator leverages filtering on top of backprojecting the data: this allows to make singularities sharper.

FBP as Numerical Implementation of Radon Reconstruction Formula



[Video credits: Samuli Siltanen]

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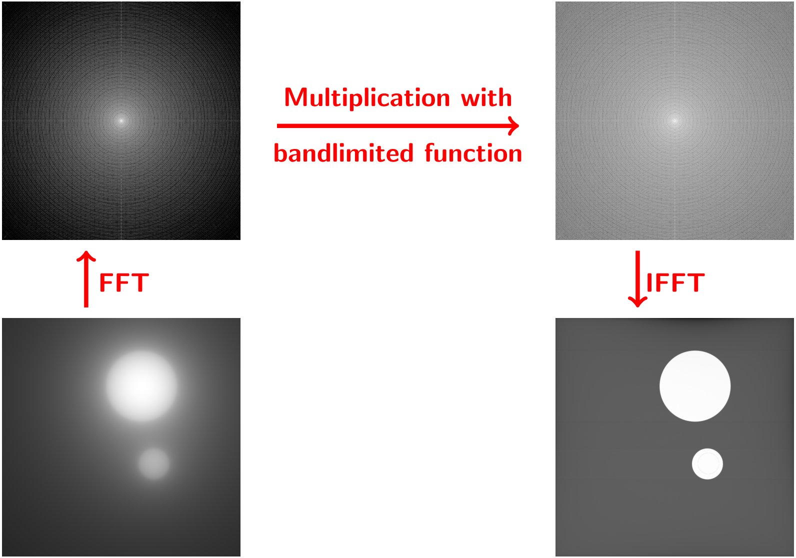
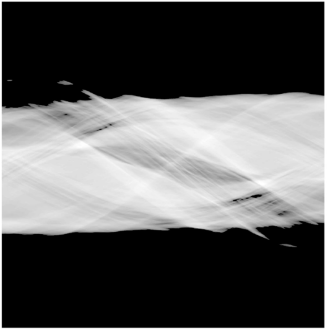
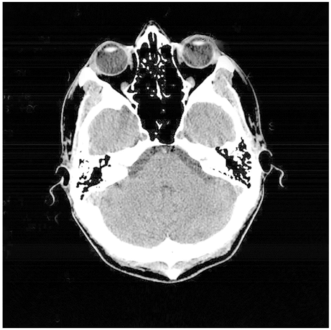


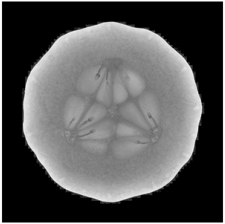
Standard FBP In Action



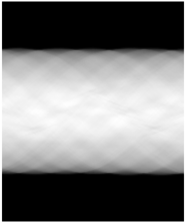


[Video credits: Samuli Siltanen]

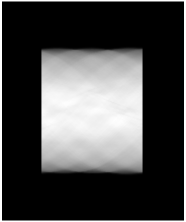
Why Do We Need Other Approaches Than FBP?



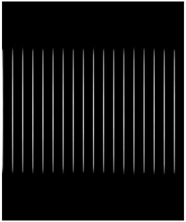
ground truth



Full angle data

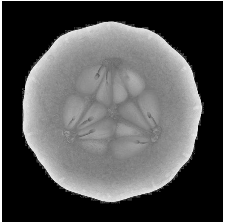


Limited-angle data

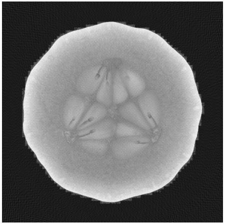


Sparse-angle data

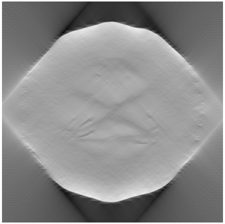
Why Do We Need Other Approaches Than FBP?



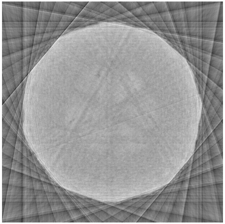
ground truth



FBP from full angle data



FBP from limited-angle data



FBP from sparse-angle data

Why Do We Need Other Approaches Than FBP?

Actually, FBP works well when:

- comprehensive projection data are available
- the target is (assumed) static

Why Do We Need Other Approaches Than FBP?

Actually, FBP works well when:

- comprehensive projection data are available
- the target is (assumed) static

In many practical tomographic applications we wish to:

- lower the X-ray radiation dose
- shorten the scanning time

$\rightsquigarrow$ **Limited Data** tomography

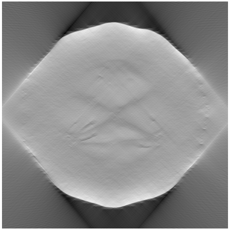
- take into account non-static target and time-dependance of measurements

$\rightsquigarrow$ **Dynamic** tomography

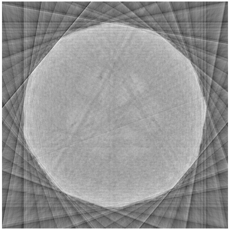
These are severely ill-posed problems and classical strategies like FBP not always suffice!

Limited Data Tomography

Notation: $\mathcal{R}_\Phi = \chi_{\Phi\times\mathbb{R}}\mathcal{R}$, with $\Phi \subset S^1$, denotes both limited data operators.



$\Phi = [\Gamma, \pi - \Gamma]$



$\Phi = [\omega_1 - \eta, \omega_1 + \eta] \cup \ldots \cup [\omega_N - \eta, \omega_N + \eta]$

- Parts of the edges, associated with specific directions, do **not** appear in the reconstruction
- Some additional edges appear, along specific lines (**streak artifacts**)
- Theoretical explanation via **microlocal analysis** (→ MiniCourse 5 by Andras Vasy)

Microlocal Analysis - Summary

Studies the singularities in functions (*e.g.*, edges in images). Some tools:

- **singular support:** the set of points x_0 near which f is not smooth;
- **wavefront set:** $\mathrm{WF}(f)$ set of pairs (x_0, ξ_0) of locations of jumps and their normal directions.

How does an operator A perturb the singularities of f?

- **Pseudo-differential op.** (ΨDO) preserve singularities: $\mathrm{WF}(Af) \subset \mathrm{WF}(f)$
- **Fourier Integral op.** (FIO) can move singularities according to a canonical relation $\mathrm{WF}(Af) \subset C(\mathrm{WF}(f))$

Key examples:

- The Radon transform $\mathcal{R}$ is a FIO (moves singularities along lines), but the normal operator $\mathcal{R}^*\mathcal{R}$ is a elliptic ΨDO
- For the limited data Radon transform $\mathcal{R}_\Phi$, theory of visibility principles (visible VS invisible singularities & streak artifacts)

Summary & Outlook

What we learned today:

- Radon transform as mathematical model of tomographic imaging
- Filtered Backprojection
- Limited data tomography

What I do not have time to talk about:

- Other geometries (fan beam)
- Extension to 3D geometries (cone beam, helical beam)
- Alternative FBP formulas and how to implement them

Up next:

- Discretization of the problem
- A splash of regularization theory of inverse problems

Some References

P.C. Hansen, J.S. Jørgensen and W.R.B. Lionheart (Eds.)
Computed Tomography: Algorithms, Insight, and Just Enough Theory
Society for Industrial and Applied Mathematics, 2021

Other classic books:

- Buzug: *Computed Tomography: From Photon Statistics to Modern Cone-Beam CT*, 2008
- Deans, *The Radon Transform and Some of Its Applications*, 1983
- Epstein, *Introduction to the mathematics of medical imaging*, 2008
- Kak & Slaney, *Principles of computerized tomographic imaging*, 1988
- Natterer, *The mathematics of computerized tomography*, 1986
- Natterer & Wübbeling, *Mathematical Methods in Image Reconstruction*, 2001