Inverse problem

Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary.
Consider the wave equation

$$\begin{aligned} &\partial_t^2 u - c^2(x)\Delta u = 0, \quad \text{in } (0,\infty)\times\Omega, \\ &\partial_\nu u|_{x\in\partial\Omega} = f, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0, \end{aligned}$$

and define the Neumann-to-Dirichlet map by $\Lambda f = u|_{x\in\partial\Omega}$.

Inverse problem. Determine c given Λ.

The Boundary Control method can be used to prove that Λ determines c.

Finite speed of propagation is given in terms of the distance function of the Riemannian manifold $(\Omega, c^{-2}(x)dx^2)$.

Control problem from the boundary

Given a function ϕ on Ω, **minimize**

$$\left\| u^f(T) - \phi \right\|^2_{L^2(\Omega)} + \alpha \, \|f\|^2_{L^2((0,T)\times\partial\Omega)}$$

subject to u^f satisfying the wave equation

$$\begin{aligned} &\partial_t^2 u - c^2(x)\Delta u = 0, \quad \text{in } (0,\infty)\times\Omega, \\ &\partial_\nu u|_{x\in\partial\Omega} = f, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0. \end{aligned}$$

Approximate controllability implies that, when $T > 0$ is large enough, the unique minimizer f_α satisfies

$$u^{f_\alpha}(T) \to \phi, \quad \alpha \to 0.$$

Blind control problem

Some control problems can be solved without knowing the speed of sound:

$$\underset{f}{\operatorname{argmin}} \left\| u^f(T) - 1 \right\|^2_{L^2(\Omega; c^{-2}dx)} + \alpha \left\| f \right\|^2_{L^2((0,T)\times\partial\Omega)} \qquad (1)$$

subject to u^f satisfying the wave equation.

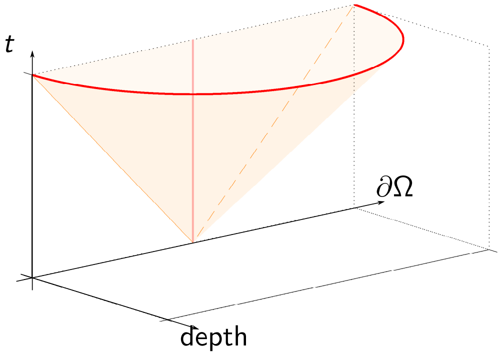
Minimization (1) is equivalent with solving $(K + \alpha)f = b$, where

$$K = J\Lambda - R\Lambda R J, \quad b(t) = T - t.$$

Here Λ is the Neumann-to-Dirichlet map, and

$$Jf(t) = \frac{1}{2}\int_t^{2T-t} f(s)ds, \quad Rf(t) = f(T-t).$$

Finite speed of propagation and supports



If the source f vanishes outside the pink line,

then u vanishes outside the cone,

and $u(T)$, for fixed T, vanishes outside the half disk inside the red arc.

$\text{supp}(f)$	$\text{supp}(u(T))$
	
	

Blind control problem with a support constraint

Let $\Gamma \subset \partial\Omega$ and $r \in (0, T)$. Consider the minimization

$$\operatorname*{argmin}_{f} \left\| u^f(T) - 1 \right\|^2_{L^2(\Omega; c^{-2}dx)} + \alpha \|f\|^2_{L^2((0,T)\times\partial\Omega)}$$

subject to u^f satisfying the wave equation and

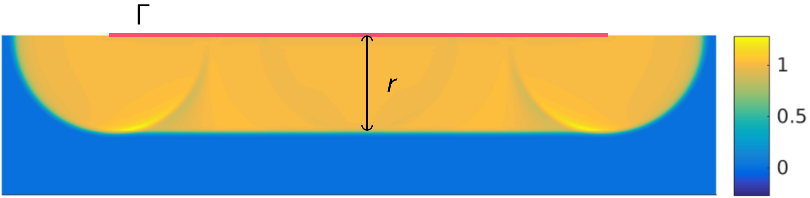
$$\operatorname{supp}(f) \subset [T - r, T] \times \Gamma.$$

Approximate controllability implies that, the unique minimizer f_α satisfies

$$\lim_{\alpha \to 0} u^{f_\alpha}(T, x) = 1_{M(\Gamma, r)}(x) := \begin{cases} 1, & x \in M(\Gamma, r) \\ 0, & \text{otherwise} \end{cases}.$$

Different support constraints

A computational solution $u^{f_\alpha}(T) \approx 1_{M(\Gamma,r)}$ to the blind control problem. Here $c(x) = 1$ for all $x \in \Omega$.



Blind control problem with a rectangle constraint $\mathrm{supp}(f) \subset [T - r, T] \times \Gamma$.

Blind control problem with a triangle constraint [de Hoop-Kepley-L.O.'16].

Localized waves in theory

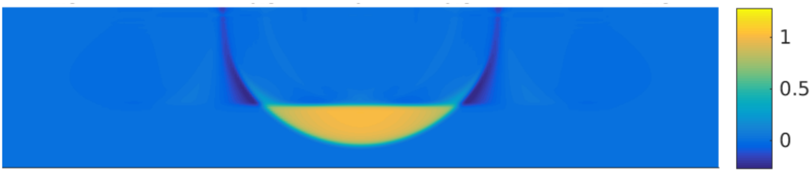
1. Blind control problem with constraint $\operatorname{supp}(f) \subset \text{rectangle} \cup \text{triangle}$

2. Blind control problem with constraint $\operatorname{supp}(f) \subset \text{rectangle}$

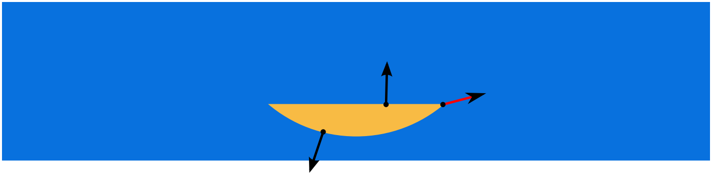
3. Difference

Localized waves in practice

The blind control problem is unstable



Microlocal explanation: the singular directions in the corners can not be produced stably using boundary sources

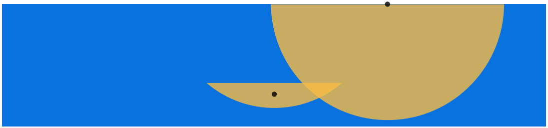


Probing the medium with localized waves

Inner products can be computed without knowing the speed of sound c:

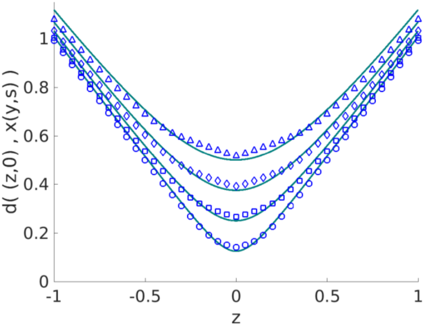
$$(u^f(T), u^h(T))_{L^2(\Omega, c^{-2}dx)} = (f, Kh)_{L^2((0,T)\times\partial\Omega)}.$$

Inner products determine the distances $d(x, y)$ where $x \in \Omega$ and $y \in \partial\Omega$.

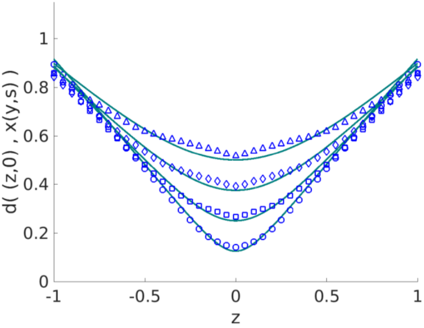


Reconstruction of distances

Distances correspond to first arrival times from "virtual point sources" (Λf with 300 Gaussian pulse sources f) [DE HOOP-KEPLEY-L.O.'16].



$c = 1$

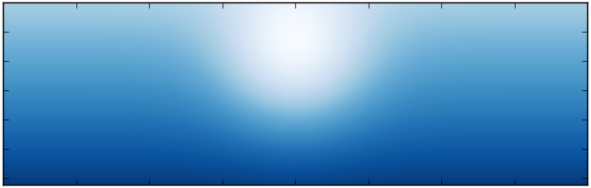


$c = 1 + \text{depth}$

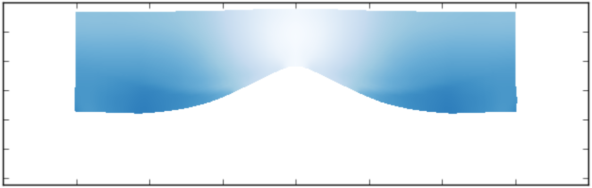
Here z is the coordinate of the receiver on the surface and the virtual point source is at the point $x = x(y, s)$ satisfying

$$y = 0 \text{ is the closest surface point to } x \quad \text{and} \quad s = d(x, y)$$

Reconstruction using the structure of $\partial_t^2 - c^2(x)\Delta$



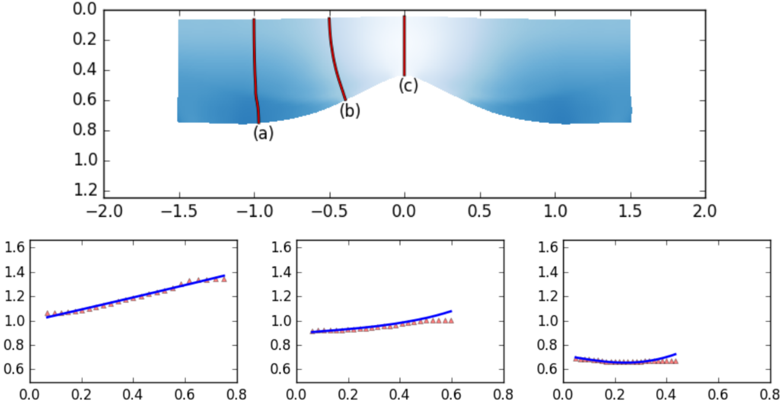
True $c(x)$



Reconstruction using the BCm[1] [DE HOOP-KEPLEY-L.O.'18]

[1] Computed from simulated measurements with 241 point like sources on the top edge

Reconstruction error

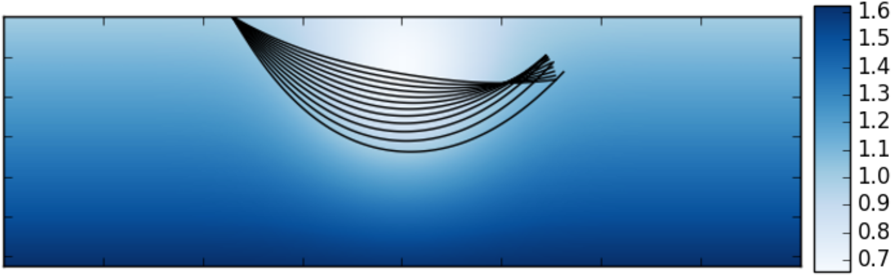


(a) (b) (c)

True c (blue) and its reconstuction (red) along a ray path.

Geodesics

Geodesics on a are the analogue of straight lines (critical points of the length functional)



Geodesics emanating from a point at the surface focus behind the lens