Projective Convergence of Measures

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Outline

Background.

Definition and basic properties.

Relation to g-measures.

Relation to random substitutions.

Final comments.

Setup

A is a finite alphabet.

We supply $A^{\mathbb{N}}$ with the topology and sigma-algebra generated by the cylinder sets.

$\mathcal{M}^+(A^{\mathbb{N}})$ is the set of all fully supported probability measures on $A^{\mathbb{N}}$ and $\mathcal{M}_T^+(A^{\mathbb{N}}) \subset \mathcal{M}^+(A^{\mathbb{N}})$ the subset containing the shift-invariant measures.

For $\phi : A^{\mathbb{N}} \to \mathbb{R}$ and $k \in \mathbb{N}$,

$$\mathrm{var}_k\phi := \max_{\boldsymbol{a} \in A^k} \sup\{|\phi(\boldsymbol{c}) - \phi(\boldsymbol{d})| : \ \boldsymbol{c}_1^k = \boldsymbol{d}_1^k = \boldsymbol{a}\}.$$

$\phi : A^{\mathbb{N}} \to \mathbb{R}$ is Hölder continuous if $\mathrm{var}_k\phi \le C\theta^k$ for some $C > 0$ and $\theta \in [0, 1)$.

Background

Preservation of g-measures under Amalgametion

A, B finite alphabets. Card$(A) >$ Card(B). $\pi : A \to B$ surjective. We extend π letterwise, $\pi : A^{\mathbb{N}} \to B^{\mathbb{N}}$.

Theorem 1 *Let $\mu \in \mathcal{M}_T^+(A^{\mathbb{N}})$ be a g-measure compatible with a Hölder continuous g-measure ψ. The induced measure $\nu = \mu \circ \pi^{-1}$ is a g-measure compatible with*

$$\boldsymbol{b} \mapsto \phi(\boldsymbol{b}) = \lim_{\ell\to\infty} \lim_{n\to\infty} \frac{\mu_\ell \circ \pi^{-1}[\boldsymbol{b}_1^n]}{\mu_\ell \circ \pi^{-1}[\boldsymbol{b}_2^n]},$$

where μ_ℓ is the Markov measure defined by the range $\ell+1$ approximantion to ψ.

In general $\mathrm{var}_\ell \phi = \mathcal{O}\left(e^{-c\sqrt{\ell}}\right)$. If ψ is locally constant, ϕ is Hölder.

The sequence $\nu_\ell := \mu_\ell \circ \pi^{-1}$ converges to $\mu \circ \pi^{-1}$ in the following sense: for each n there exists a sequence $\{\epsilon_{\ell,n}\}_{\ell\in\mathbb{N}}$ such that $\lim_{\ell\to\infty} \epsilon_{\ell,n} = \lim_{n\to\infty} \epsilon_{\ell,\ell^2} = 0$ and

$$\exp(-\epsilon_{\ell,n}) \leq \frac{\nu_\ell[\boldsymbol{b}]}{\nu[\boldsymbol{b}]} \leq \exp(\pm\epsilon_{\ell,n})$$

for all $\boldsymbol{b} \in B^n$.

On the one hand ν_ℓ is a g-measure compatible with the Hölder continuous function $\boldsymbol{b} \mapsto \phi_\ell(\boldsymbol{b}) = \lim_{n\to\infty} \nu_\ell[\boldsymbol{b}_1^{n+1}]/\nu_\ell[\boldsymbol{b}_2^{n+1}]$.

Finally, for any $\boldsymbol{b} \in B^{\mathbb{N}}$,

$$\left|\phi_\ell(\boldsymbol{b}) - \frac{\nu[\boldsymbol{b}_1^{\ell^2+1}]}{\nu[\boldsymbol{b}_2^{\ell^2+1}]}\right| \leq 2\,\epsilon_{\ell,\ell^2} + C\,\ell^2\,\theta^\ell. \qquad (1)$$

for some $C > 0$ and $\theta \in [0,1)$.

The projective distance

Let $\mathcal{M}^+(A^{\mathbb{N}})$ be the set of fully-supported Borel probability measures on $A^{\mathbb{N}}$. We define $\rho : \mathcal{M}^+(A^{\mathbb{N}}) \times \mathcal{M}^+(A^{\mathbb{N}}) \to \mathbb{R}^+$ by

$$\rho(\mu, \nu) = \sup_{n \in \mathbb{N}} \max_{\boldsymbol{a} \in A^n} \frac{1}{n} \left| \log \frac{\mu[\boldsymbol{a}]}{\nu[\boldsymbol{a}]} \right| . \tag{2}$$

ρ defines a distance which we call projective distance.

Theorem 2 $\mathcal{M}^+(A^{\mathbb{N}})$ *is complete with respect to* ρ*.*

This follows from the fact that the vague topology is weaker than the one induced by ρ.

$\lambda \in \mathcal{M}((A \times A)^{\mathbb{N}})$ such that

$$\sum_{\boldsymbol{b} \in A^n} \lambda[\boldsymbol{a} \times \boldsymbol{b}] = \mu[\boldsymbol{a}], \ \sum_{\boldsymbol{a} \in A^n} \lambda[\boldsymbol{a} \times \boldsymbol{b}] = \nu[\boldsymbol{b}],$$

is a coupling between μ and $\nu \in \mathcal{M}(A^{\mathbb{N}})$. $J(\mu,\nu) \subset \mathcal{M}((A \times A)^{\mathbb{N}})$ denote the set of all couplings between μ and ν.

Ornstein's $\bar{d}$-distance is given by

$$\bar{d}(\mu,\nu) = \inf_{\lambda \in J(\mu,\nu)} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(T^{-k} \bar{\Delta}), \tag{3}$$

where $\bar{\Delta} = \{ab \in A \times A : a \neq b\}$. $\bar{d}$ makes $\mathcal{M}(A^{\mathbb{N}})$ complete but non-separable.

Theorem 3 $\mathcal{M}^+(A^{\mathbb{N}})$ *is non-separable with respect to* ρ.

Fix $\boldsymbol{x} \in \{0,1\}^{\mathbb{N}}$, and for each $n \in \mathbb{N}$ and $\boldsymbol{a} \in \{0,1\}^n$ let

$$q(\boldsymbol{a}) = \max\{1 \le k \le n : \ \boldsymbol{a}_1^k = \boldsymbol{x}_1^k\} + 1.$$

Fix $\alpha = e^{1/2}/(2 - e^{1/2})$ and let $\nu_{\boldsymbol{x}} \in \mathcal{M}^+(\{0,1\}^{\mathbb{N}})$ be given by

$$\nu_{\boldsymbol{x}}[\boldsymbol{a}] = \begin{cases} \alpha^n (1+\alpha)^{-n} & \text{if } \boldsymbol{a} = \boldsymbol{x}_1^n, \\ \alpha^{q(\boldsymbol{a})-1}(1+\alpha)^{-q(\boldsymbol{a})} 2^{q(\boldsymbol{a})-n} & \text{if } \boldsymbol{a} \neq \boldsymbol{x}_1^n, \end{cases} \tag{4}$$

for all n and $\boldsymbol{a} \in \{0,1\}^n$.

The collection $\{\mu_{\boldsymbol{x}} \in \mathcal{M}^+(\{0,1\}^{\mathbb{N}}) : \ \boldsymbol{x} \in \{0,1\}^{\mathbb{N}}\}$, is such that $\rho(\mu_{\boldsymbol{x}}, \mu_{\boldsymbol{y}}) > 1/2$ whenever $\boldsymbol{x} \neq \boldsymbol{y}$.

Theorem 4 *There exists a sequence* $\{\mu_p \in \mathcal{M}^+(A^{\mathbb{N}})\}_{p\in\mathbb{N}}$ *converging in* $\bar{d}$*-distance, but not in the projective distance.*

Fix $\boldsymbol{x} \in \{0,1\}^{\mathbb{N}}$ and for each $p \in \mathbb{N}$ let $\boldsymbol{x}_p \in \{0,1\}^{\mathbb{N}}$ be such that

$$(\boldsymbol{x}_p)_k = \begin{cases} 1 - x_k & \text{if } k \in p\,\mathbb{N} + 1, \\ x_k & \text{if } k \notin p\,\mathbb{N} + 1. \end{cases}$$

And define $\mu_{\boldsymbol{x}_p}$ and $\mu_{\boldsymbol{x}}$ as in Equation (4).

It can be proved that $\mu_{\boldsymbol{x}} = \lim_{p\to\infty} \mu_{\boldsymbol{x}_p}$ in $\bar{d}$-distance. On the other hand, Theorem 3 ensures that $\rho(\mu_{\boldsymbol{x}_p}, \mu_{\boldsymbol{x}_{p'}}) > 1/2$ for all $p \neq p'$.

Relation to g-measures

A continuous function $g : A^{\mathbb{N}} \to (0,1)$ satisfying $\sum_{x_1} g(\boldsymbol{x}) = 1$ is a g-function. A compatible g-measure is any shift-invariant probability measure μ satisfying $\lim_{n\to\infty} \mu(\boldsymbol{x}_1 = a_1 | \boldsymbol{x}_2^n = \boldsymbol{a}_2^n) = g(\boldsymbol{a})$, for all $\boldsymbol{a} \in A^{\mathbb{N}}$.

Theorem 5 *Let g be a g-function and $\mu \in \mathcal{M}(A^{\mathbb{N}})$ a compatible g-measure. For each $\ell \in \mathbb{N}$ let $\mu_\ell \in \mathcal{M}(A^{\mathbb{N}})$ be the canonical ℓ-step Markov approximation. Then $\mu_\ell \to \mu$ as $\ell \to \infty$ in the projective distance. Furthermore, $\rho(\mu_\ell, \mu) \leq \mathrm{var}_\ell \log \circ g$.*

Related to non-uniqueness

Corollary 1 *There are sequences converging in the projective distances, but not in the $\bar{d}$-distance.*

All instances of non-uniqueness provide such a sequence.

Related to uniqueness

Theorem 6 *Assume g admits a unique g-measure μ and suppose that $\{\mu_p\}_{p\in\mathbb{N}}$ is a sequence of ergodic measures converging to μ in the projective distance, then*

$$\lim_{p\to\infty} h(\mu_p) = h(\mu) \equiv -\int \log \circ g \; d\mu.$$

Theorem 7 *Let $\{g_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of g-functions converging to g in the sup-norm, and such that for each $\ell \in \mathbb{N}$ the function g_ℓ is locally constant of range $\ell+1$. If*

$$\lim_{\ell\to\infty} \left\|\log\left(\frac{g}{g_\ell}\right)\right\| \exp\left(\sum_{k=0}^{\ell} \mathrm{var}_k \log \circ g_\ell\right) = 0,$$

then the sequences $\{\mu_\ell\}_{\ell\in\mathbb{N}}$, where μ_ℓ is the corresponding Markov approximation, converges in projective distance. Furthermore, the limit measure is the unique measure compatible with g.

Relation to random substitutions

Σ is a finite collection of substitutions. $\sigma : A \to A^L$, for all $\sigma \in \Sigma$.

Substitutions are extended letterwise $\boldsymbol{\sigma} : A^k \to A^{Lk}$ for $\boldsymbol{\sigma} \in \Sigma^k$.

$\nu \in \mathcal{M}_T(\Sigma^{\mathbb{N}})$ defines the random substitution

$S : \mathcal{M}_T(A^{\mathbb{N}}) \to \mathcal{M}_T(A^{\mathbb{N}})$ as follows.

In general $S : \mathcal{M}(A^{\mathbb{N}}) \to \mathcal{M}(A^{\mathbb{N}})$ is such that

$$S\mu[\boldsymbol{a}] = \sum_{[a] \sqsubseteq \boldsymbol{\sigma}(\boldsymbol{b})} \nu[\boldsymbol{\sigma}]\mu[\boldsymbol{b}]. \tag{5}$$

Here $\boldsymbol{a} \sqsubseteq \boldsymbol{c}$ stands for "$\boldsymbol{a}$ is prefix of $\boldsymbol{c}$".

A realization $\boldsymbol{b} \in A^{\mathbb{N}}$ of $\mathcal{S}\mu$ is a sequences obtaned for a μ-randomly chosen sequences $\boldsymbol{a} \in A^{\mathbb{N}}$, subject to a ν-randomly chose sequences of substitutions $\boldsymbol{\sigma} \in \Sigma^{\mathbb{N}}$, $\boldsymbol{b} := \boldsymbol{\sigma}_1(\boldsymbol{a}_1) \cdots \boldsymbol{\sigma}_n(\boldsymbol{a}_n) \cdots$

Decay of correlations

The measure $\mu \in \mathcal{M}(A^{\mathbb{N}})$ has decay of correlations if

$$\lim_{n\to\infty} |\mu(\boldsymbol{x}_n = \boldsymbol{a}_n,\, \boldsymbol{x}_{n+1}^{n+N} = \boldsymbol{a}_{n+1}^{n+N}) - \mu(\boldsymbol{x}_n = \boldsymbol{a}_n,) \times \mu(\boldsymbol{x}_{n+1}^{n+N} = \boldsymbol{a}_{n+1}^{n+N})| = 0.$$

Projective convergence

Theorem 8 *If $\nu \in \mathcal{M}(\Sigma^{\mathbb{N}})$ is a product measure and $\bigcup_{\sigma\in\Sigma, a\in A} \sigma(a) = A^L$, then there exists $\mu^* \in \mathcal{M}^+(A^{\mathbb{N}})$ such that, for any $\mu \in \mathcal{M}^+(A^{\mathbb{N}})$,*

$$\mu^* = \lim_{n\to\infty} S^n \mu$$

exponentially fast in projective distance. Furthermore μ^* has decay of correlations.

The result relies on three facts.

Fact 1

If $\mu \in \mathcal{M}(A^{\mathbb{N}})$ is a product measure, then $S^n\mu$ is a non-homogeneous Markov measure of range $L_n := L^n$.

Fact 2

Theorem 9 *If $\{\mu^{(k)}\}_{k\in\mathbb{N}}$ is a sequence of non-homogeneous Markov measures in $\mathcal{M}_T^+(A^{\mathbb{N}})$, $\mu^{(k)}$ has range L_k and*

$$\sum_{k=1}^{\infty} \rho\left(\mu^{(k)}, \mu^{(k+1)}\right) L_{k+1} < \infty,$$

then $G = \{g_n(\boldsymbol{a}) := \lim_{N\to\infty} \mu^{(N)}(\boldsymbol{x}_n = \boldsymbol{a}_n | \boldsymbol{x}_{n+1}^{L_N} = \boldsymbol{a}_{n+1}^{L_N})\}$ defines a system of transtion probabilities and the projective limit $\mu^* := \lim_{k\to\infty} \mu^{(k)} \in \mathcal{M}_T^+(A^{\mathbb{N}})$ is a process consistent with the G. The limit process has decay of correlations.

Fact 3

Theorem 10 *If* $\nu \in \mathcal{M}_T(\Sigma^{\mathbb{N}})$ *is a product measure and* $\bigcup_{\sigma\in\Sigma, a\in A} \sigma(a) = A^L$*, then there exists* $\tau \in [0,1)$ *such that,*

$$\rho(\mathcal{S}\mu, \mathcal{S}\mu') \leq \frac{\tau}{L}\rho(\mu, \mu')$$

for all $\mu, \mu' \in \mathcal{M}_T^{+}(A^{\mathbb{N}})$.

Final comments

Phase transition should be detectable through a criterion of the kind

$$\lim_{\ell\to\infty} \left\|\log\left(\frac{g}{g_\ell}\right)\right\| \exp\left(\sum_{k=0}^{\ell} \mathrm{var}_k \log \circ g_\ell\right) = 0.$$

we have for uniqueness.

The techniques so far developped are unsufficient to treat the case of variable length substitution.

In the general case we can prove convergence in vage topology towards a unique fixed point. We don't know about the g character of the limit measure. If so, the associated g-function would not be Hölder.

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The expansion-modification system can be described as follows. Consider the random substitution

$$x \mapsto \begin{cases} 1-x & \text{with probability } p, \\ xx & \text{with probability } 1-p, \end{cases}$$

The two-sites correlation function

$$C_p(n) := \int_X x_0 x_n \, d\mu_p(\mathbf{x}) - \left(\int_X x_0 \, d\mu_p(\mathbf{x}) \right) \left(\int_X x_n \, d\mu_p(\mathbf{x}) \right),$$

satisfy $A_p \, n^{-\beta_p} \le C_p(n) \le B_p \, n^{-\beta_p}$ for constants $A_p \le 1 \le B_p$.

A system of transition probabilities is a collection $G = \{g_n\}_{n\in\mathbb{N}}$ of g-functions.

A process consistent with G is a probability measure $\mu \in \mathcal{M}(A^{\mathbb{N}})$ such that $\lim_{N\to\infty} \mu(\boldsymbol{x}_n = \boldsymbol{a}_n | \boldsymbol{x}_{n+1}^{n+N} = \boldsymbol{a}_{n+1}^{n+N}) = g_n(T^{n-1}\boldsymbol{a})$, for all $\boldsymbol{a} \in A^{\mathbb{N}}$. Here T stands for the left shift.