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Ball Box System in $\mathbb{Z}$

Joint work with Chi Nguyen, Leonardo Rolla and Minmin Wang.

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There is a box at each integer $x \in \mathbb{Z}$.

Ball configuration $\eta \in \{0,1\}^{\mathbb{Z}}$: 0 = empty box, 1 = ball.

Finite number of balls:

Empty carrier starts to the left of leftmost ball, and visits boxes from left to right.

Let $T\eta$: configuration after the carried visited all boxes.

```
0  0  1  0  1  1  0  0  0  1  1  1  0  1  0  0  0  0  0      η
  0  0  1  0  1  2  1  0  0  1  2  3  2  3  2  1  0  0  0    carrier
0  0  0  1  0  0  1  1  0  0  0  0  1  0  1  1  1  0  0      Tη
```

Dynamical system, nothing random in the evolution.

Ball-Box-System introduced by Takahashi-Satsuma (1990)

KdV equation

BBS has solitons, a phenomenon present in the Korteweg & de Vries (KdV) differential equation for $u(r,t) \in \mathbb{R}^+$, $r \in \mathbb{R}$, $t \in \mathbb{R}^+$ given by

$$\dot{u} = u''' + u\,u' \tag{1}$$

For the relation between BBS and KdV see Tokihiro et al , Takahashi and Matsukidaira and Kato, Satoshi and Zuk .

But the way is not easy: From the paper of Tokihiro et al:

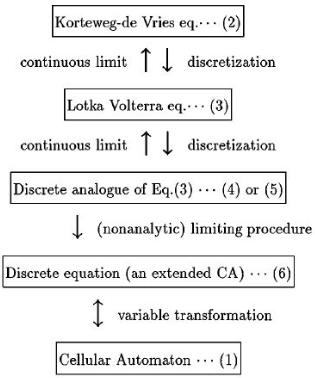


FIG. 2. The route from the KdV equation [Eq. (2)] to the integrable cellular automaton [Eq. (1)]. The numbers on the right side of the boxes correspond to those of equations in the text.

Here BBS with infinitely many balls.

If there are "more balls than empty boxes", the carrier gets an increasing number of balls and the result is just a global flipping: $T\eta(x) = 1 - \eta(x)$. Hence consider configurations with "upper density less than half":

$$\mathcal{X}_{\frac{1}{2}} := \Big\{\eta \in \{0,1\}^{\mathbb{Z}} : \limsup_{n\to\pm\infty} \frac{1}{n}\Big|\sum_{x=0}^{n} \eta(x)\Big| < 1/2\Big\},$$

If $\eta \in \mathcal{X}_{\frac{1}{2}}$, then the carrier load after visiting box x defined by

$$C(x,\eta) := \Big(\sup_{z\le x} \sum_{y=z}^{x} (2\eta(y) - 1)\Big)^{+}$$

is bounded and $(T\eta)(x) := (C(x-1,\eta) - C(x,\eta))^{+}$ is well defined.

Since $T\eta$ may not belong to $\mathcal{X}_{\frac{1}{2}}$ we define

$$\mathcal{X} := \Big\{\eta \in \mathcal{X}_{\frac{1}{2}} : \limsup_{y\to\pm\infty}\Big|\frac{1}{y}\sum_{x=0}^{y} C(x,\eta)\Big| < \infty\Big\}.$$

We show that if $\eta \in \mathcal{X}$, then $T^n\eta \in \mathcal{X}_{\frac{1}{2}}$ and the dynamics is well defined for all times.

The operator $T : \mathcal{X} \to \mathcal{X}$ induces operators in the space of bounded functions: $(Tf)(\eta) = f(T\eta)$ and of measures: $(\mu T)f = \mu(Tf)$.

Invariant measures A measure is invariant if $\mu T = \mu$.

Let ν_λ product with density λ (iid Bernoulli(λ)).

Theorem 1 ν_λ *is invariant for* T *for all* $\lambda < \frac{1}{2}$.

Proof Application of Burke Theorem. Under iid Bernoulli(λ), carrier performs a reflecting random walk on $\{0, 1, 2, \dots\}$ with

$p(\ell, \ell + 1) = \lambda$, $p(\ell + 1, \ell) = 1 - \lambda$ for $\ell \geq 0$ and $p(0, 0) = 1 - \lambda$.

Since $\lambda < 1/2$, Geometric$(\lambda/(1 - \lambda))$ is reversible.

By reversibility, up-jumps $\sim$ reflected down-jumps.

up-jumps $=$ balls in η, down-jumps $=$ balls in $T\eta$.

iid Bernoulli reflecting invariant implies down jumps $\sim$ iid Bernoulli(λ).
$\square$

Goal Characterize the set of shift-ergodic invariant measures for T.

Conserved quantities Number of balls is conserved for $\eta \in \mathcal{X}$.

Number of $\eta(x)(1-\eta(x+1)$ is conserved.

We call *k-pode* a set of k successive ones followed by k zeroes in the middle of zeroes. k-podes are conserved and travel at speed k:

```
000001110000000000000
000000001110000000000
000000000001110000000
000000000000001110000
```

Isolated k-podes travel at speed k and conserve the distances:

```
000001110000000000000000001110000000000000
000000001110000000000000000001110000000000
000000000001110000000000000000001110000000
000000000000001110000000000000000001110000
```

k-podes are conserved even when interacting with m-podes:

```
000001110000000100000000000011100000000000000000000000
000000001110000010000000000000011100000000000000000000
000000000001110001000000000000000111000000000000000000
000000000000001110100000000000000000011100000000000000
000000000000000001011100000000000000000011100000000000
000000000000000000100011100000000000000000011100000000
```

Method of TS to identify k-podes (basic sequences) when they are in the same cycle.

TS algorithm to identify k-podes when there is a finite number of balls

Run: sequence of successive zeroes between two ones or sequence of ones between two zeroes.

Start from the left and look at the runs from left to right.

When a run has length $k < m$, the lenght of the following run, then the k elements of the short run and the first k elements of the long run are called k-pode.

Identified k-podes are ignored by the algorithm.

Repeat the procedure until all multipodes have been identified.

Records Given $\eta \in \mathcal{X}$, we say that there is a *record* at x if

$$\sum_{y=x-z}^{x} (2\eta(y) - 1) < 0, \qquad \text{for all } z \geq 0.$$

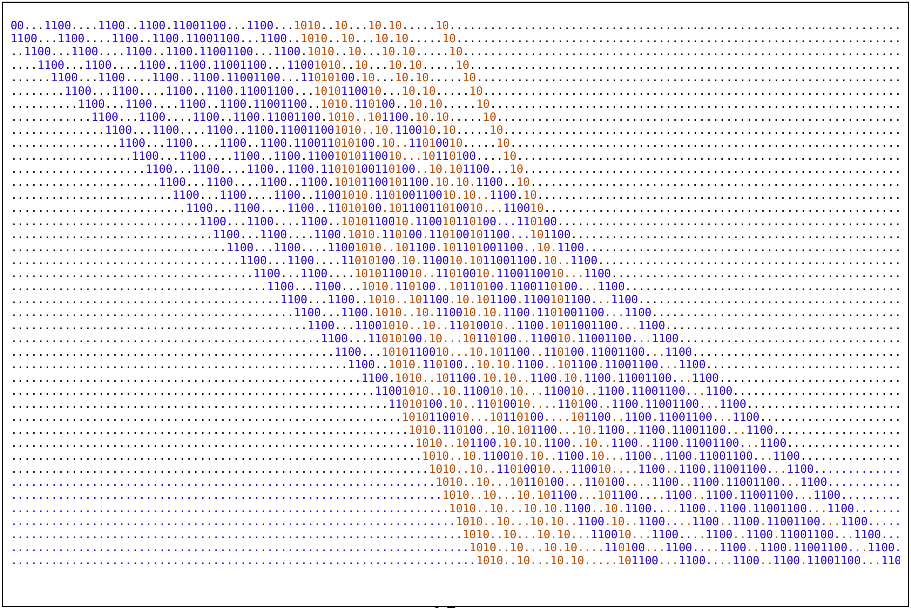
Trajectory of a nearest neighbor walk $(\xi(x) : x \in \mathbb{Z})$ with increments

$$\xi(x) - \xi(x-1) = 2\eta(x) - 1$$

Record at x for the walk if $\xi(x) < \xi(x-z)$ for all $z > 0$.

Use dots for records:

```
.....111000....10...........111000....................
........111000..10.............111000.................
...........11100010..............111000...............
..............11101000...............111000...........
.................10111000...............111000........
..................10..111000...............111000.....
```



Effective distance between successive k-podes

Number of records between them when isolated from m-podes for $m > k$.

```
.....111000....10...10.....11011000......................
........111000..10...10......10.111000...................
...........11100010...10......10...111000................
..............11101000.10......10.....111000.............
.................1011100100.....10.......111000..........
..................10..11011000...10..........111000.....
...................10...10.111000.10...........111000..
```

For $m > k$:

• When m-pode overpasses k-pode, the k-pode gains $2(m-k)$ records with respect to its natural trajectory.

• k-podes do not affect effective distance of m-podes.

The tagged record process System as seen from a tagged record.

Configurations with a record at the origin:

$$\mathcal{X}^o := \eta \in \mathcal{X} : \text{there is a record at } 0\}$$

Take $\eta \in \mathcal{X}^o$,

$$r_t(\eta, i) := \text{position in } T^t\eta \text{ of } i\text{th record of } \eta$$

```
........111000..10...10..x...111000................
...........11100010...10.x.....111000..............
..............11101000.10x........111000...........
...............x.1011100100..........111000........
...............x..10..11011000..........111000.....
...............x...10...10.111000..........111000..
```

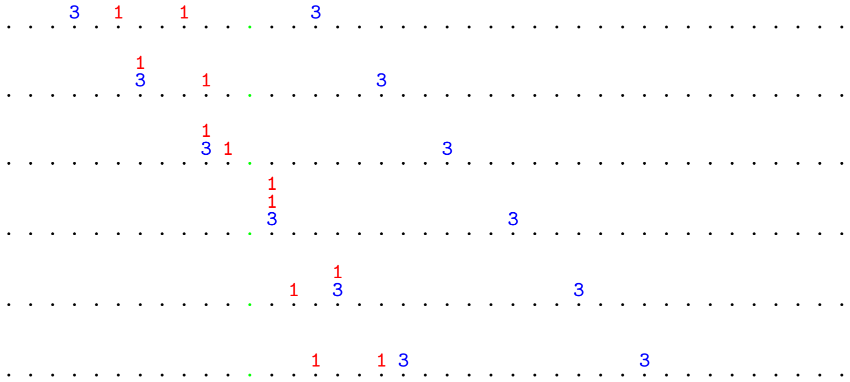
x is the tagged record $r_t(\eta, 0)$

For $\eta \in \mathcal{X}^o$ system at time t as seen from the tagged record:

$$\hat{T}^t \eta = \theta_{r_t(\eta,0)} T^t \eta$$

```
....111000..10...10..x...111000..........................
.......11100010...10.x.....111000........................
..........11101000.10x........111000.....................
.....................x.1011100100..........111000........
.....................x..10..11011000..........111000.....
.....................x...10...10.111000..........111000..
```

Fuzzy representation



3-pode motion is predictable from the single time fuzzy configuration.

The 1-pode evolution can be reconstructed from the last 3 steps.

Shift-ergodicity Shift $(\theta_x \eta)(y) = \eta(y - x)$.

A shift-stationary μ is *ergodic* if for all measurable $f : \mathcal{X} \to \mathbb{R}$,

$$\theta_x f = f \quad \text{implies } f \text{ is constant } \mu\text{-a.s.}$$

Equivalent to Cesaro asymptotic independence:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=0}^{n+1} f(\eta) g(\theta_x \eta) = \mu f \, \mu g.$$

for continuous f and g.

Record-shift-ergodicity μ is *record-stationary* if $\mu(\mathcal{X}^o) = 1$ and

$$\int f(\theta_{r(\eta,i)}\eta)\mu(d\eta) = \mu f$$

for all record i and measurable f. (point stationarity, Thorisson).

A record-stationary μ is *record-ergodic* if for all $f : \mathcal{X}^o \to \mathbb{R}$

$$\theta_{r(\eta,i)} f(\eta) = f(\eta), \text{ for all } \eta, i \quad \text{implies } f \text{ is constant } \mu\text{-a.s.}$$

Equivalent to Cesaro asymptotic independence along records:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n+1} f(\eta) g(\theta_{r(\eta,i)}\eta) = \mu f \mu g, \qquad \mu\text{a.s.}$$

for continuous f and g.

Palm measure μ shift-stationary.

$\hat{\mu} := \mu$ conditioned to record at origin:

$$\hat{\mu} f = \frac{1}{\mu(\mathcal{X}^o)} \int f(\eta) \mathbf{1}\{\eta \in \mathcal{X}^o\} \mu(d\eta).$$

Proposition 2 *(a) μ shift-stationary iff $\hat{\mu}$ record-shift stationary.*

(b) μ ergodic iff $\hat{\mu}$ record-ergodic.

Theorem 3 (Harris, Port& Stone) *Let μ be shift-stationary. Then,*

$$\hat{\mu}\hat{T} = (\mu T)\hat{}$$

In particular $\mu T = \mu$ if and only if $\hat{\mu}\hat{T} = \hat{\mu}$.

Proof uses that T commutes with shift: $T\theta_x = \theta_x T$, for all $x \in \mathbb{Z}$.

Multipode decomposition

Family of operators $M_k : \mathcal{X}^o \to \mathcal{X}^o$.

$M_k\eta$ has only k-podes at its *effective distance*.

```
..........11101000.10x..10...111000.10..............     η
.............10...10x..10......10...................   M_1η
...................x................................   M_2η
.............111000.x....111000.....................   M_3η
                             ...                         ...
```

Effective distance between successive k-podes computed by evolving the piece of configuration containing the cycles of both k-podes and everything between them.

Define $M : \mathcal{X}^o \to (\mathcal{X}^o)^{\mathbb{N}}$ given by $M : \eta \mapsto (M_k\eta : k \geq 1)$.

Proposition 4 *M is one-to-one.*

Shift-ergodic invariant measures

Our **main result:**

Theorem 5 *Let μ be shift-ergodic invariant with $\lambda < 1/4$ and let $\hat{\mu}$ be its Palm measure. Then,*

$$\hat{\mu}M = \bigotimes_{k\geq 1} \hat{\mu}M_k.$$

That is, if η has law $\hat{\mu}$,

$(M_k\eta : k \geq 1)$ is a family of independent configurations.

Recall that $M_k\eta$ contains only k-podes.

Evolution of decomposed configurations

Multipode decomposition of the configuration at time t consists of distinct translations of the decomposition of the initial configuration

Let $y_t(\eta, k, i)$ be the increment in records of the first k-pode of $\hat{T}^t\eta$ to the right of $r(\hat{T}^t\eta, i)$ in the time interval $[0, t]$.

Theorem 6 *For $\eta \in \mathcal{X}^o$ and $k \geq 1$ we have*

$$M_k \hat{T}^t \eta = M_k \theta_{-y_t(\eta,k)} \eta$$

Let ρ_k be the mean number of k-podes per record under $\hat{\mu}$. Since a k-pode occupies $2k$ sites in a cycle, we have that

$a := 2\sum_k k\rho_k$ is the mean cycle size under $\hat{\mu}$.

By ergodicity, the density of balls $\lambda = \frac{a/2}{a+1}$. Hence,

$$a = \sum_k 2k\rho_k = \frac{2\lambda}{1-2\lambda} < \infty.$$

Proposition 7 *μ shift-ergodic with $\lambda < 1/4$ and $\hat{\mu}$ its Palm measure. Assume $\rho_k > 0$, $\rho_\ell > 0$ and $\ell < k$. Then*

$$0 < c \le \underline{\lim}_{t\to\infty} \frac{y_t(\eta,k,i) - y_t(\eta,\ell,i)}{t} \le \overline{\lim}_{t\to\infty} \frac{y_t(\eta,k,i) - y_t(\eta,\ell,i)}{t} \le k - \ell, \tag{2}$$

for $\hat{\mu}$ almost all η; $c = c(k,\ell)$.

For the moment the inequality $0 < c$ holds under $\lambda < 1/4$.

Proof of Theorem 5

$$\hat{\mu}(M_k f\, M_m g) = \int \hat{\mu}(d\eta) f(M_k \hat{T}^t \eta)\, g(M_m \hat{T}^t \eta) \quad \text{(invariance of } \hat{\mu}\text{)}$$

$$= \int \hat{\mu}(d\eta) f(M_k \theta_{-y_t(\eta,k,0)} \eta)\, g(M_m \theta_{-y_t(\eta,m,0)} \eta) \quad \text{(Thm 6)}$$

$$= \int \hat{\mu}(d\eta) f(M_k \eta)\, g(M_m \theta_{-y_t(\eta,m,0)+y_t(\eta,k,0)} \eta) \quad \text{(stationarity)}$$

Hence,

$$\hat{\mu}(M_k f\, M_m g) = \int \hat{\mu}(d\eta)\frac{1}{t}\sum_{s=0}^{t} f(M_k\eta)\, g(M_m\theta_{-y_s(\eta,m,0)+y_s(\eta,k,0)}\eta)$$

$$\underset{t\to\infty}{\to} \int \hat{\mu}(d\eta) f(M_k\eta) \int \hat{\mu}(d\eta) g(M_m\eta) \quad \text{(ergodicity of } \hat{\mu}\text{)}$$

$$= \hat{\mu}M_k f\, \hat{\mu}M_m g$$

The limit uses that for large t,

$$y_t(\eta, m, 0) - y_t(\eta, k, 0) \;\geq\; tc(k, m) \;\to\; \infty. \quad \text{(Proposition 7, b)} \;\square$$

Speed of k-podes

Proposition 8 *μ shift-ergodic and $\hat{\mu}$ its Palm measure. Assume $\rho_k > 0$ and $\sum_{m\geq 1} m^2 \rho_m < \infty$. Then the following limits exist*

$$\lim_{t\to\infty} \frac{y_t(\eta,k,i)}{t} = v_k < \infty \tag{3}$$

for $\hat{\mu}$ almost all η. The limits are the unique solution of the system

$$v_k = k + \sum_{m=k+1}^{\infty} 2(m-k)(v_m - v_k)\rho_m, \quad k \geq 1. \tag{4}$$

and satisfy the following bounds:

$$\frac{\ell}{c_\ell} + 2\sum_{m>\ell} m(m-\ell)\frac{\rho_m}{c_m} \leq v_\ell \leq \frac{\ell}{c_\ell} + 2c_\ell \sum_{m>\ell} m(m-\ell)\frac{\rho_m}{c_m} \tag{5}$$

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