Furstenberg's conjecture on intersections of Cantor sets, and self-similar measures

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Base p expansions

Let $p \in \mathbb{N}_{\geq 2}$. Every point x has an expansion to base p:

$$x = 0.x_1 x_2 \ldots = \sum_{n=1}^{\infty} x_n p^{-n}, \quad x_i \in \{0, 1, \ldots, p-1\}.$$

Basic facts:

1. All but countably many (rational) points have a unique expansion; the remaining ones have two expansions.
2. A point is rational if and only if the expansion is eventually periodic.
3. Expansions in bases p^n and p^k are "almost the same" (look at base p in blocks of length n and k).

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Multiplication by p

Definition

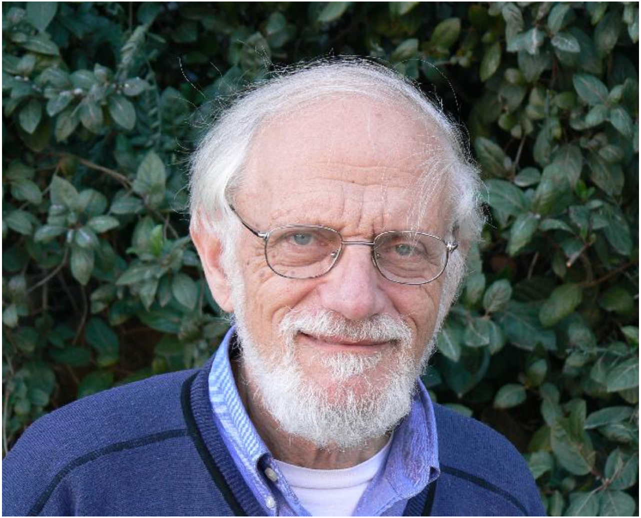
For $p \in \mathbb{N}_{\geq 2}$, let

$$T_p = px \bmod 1$$

be multiplication by p on the circle.

Symbolically, $T_p x$ corresponds to shifting the p-ary expansion x: there is a factor map, which is one-to-one outside of the countably many points with two p-ary expansions.

Multiplying by 2 and by 3: the founding father



Some of the areas that Furstenberg initiated

1. Ergodic theoretic methods in combinatorics (ergodic proof of Szemerédi's Theorem,...).
2. Products of random matrices, non-commutative ergodic theory (simplicity of Lyapunov exponents, ...).
3. Unique ergodicity of horocycle flow, toral maps, ...
4. Disjointness of dynamical systems.
5. $\times 2$, $\times 3$, rigidity of higher order actions.
6. Fractal geometry $\cap$ ergodic theory (CP-processes, ...).

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Expansions in different bases

Principle (Furstenberg)

Expansions in bases 2 and 3 have no common structure.
More generally, this holds for bases p and q which are not powers of a common integer or, equivalently, $\log p / \log q$ is irrational.

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Invariant sets

Definition

A set $A \subset [0,1)$ is T_p-invariant if $T_p(A) \subset A$. That is, shifting the p-ary expansion of a point in A gives another point in A.

- If p and q are coprime, then $\{0, 1/q, \ldots, (q-1)/q\}$ is T_p-invariant.
- $[0,1)$ is T_p-invariant.
- Let $D \subset \{0, 1, \ldots, p-1\}$. The set $A = A_{p,D}$ of points whose base p-expansion has only digits from D is T_p-invariant. We call it a p-Cantor set. Example: the middle-thirds Cantor set.
- There is a wild abundance of invariant sets and no classification or description is possible.

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Invariant sets and shared structure

Principle (Furstenberg, slightly more concrete version)

If A, B are closed and invariant under T_2, T_3 respectively, then A and B have no common structure.

Theorem (Furstenberg (1967))

If A is jointly invariant under T_2 and T_3, then A is either finite or the whole circle $[0, 1)$.

Remarks

- *The theorem is a weak confirmation of the principle since the set A and itself certainly have a lot of common structure!*
- *One should think of finite sets and the whole circle as sets "without structure".*

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A corollary in terms of orbits

Observation

- *If x is rational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is infinite.*
- *If x es irrational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is infinite (and its closure is invariant under T_2 and T_3).*

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"The" $\times 2$, $\times 3$ Furstenberg conjecture

Definition

A Borel probability measure μ on $[0,1)$ is T_p-invariant if

$$\mu(B) = \mu(T_p^{-1}B) \quad \text{for all Borel sets } B.$$

Conjecture (Furstenberg 1967)

If μ is T_2 and T_3 invariant, then μ is a convex combination of Lebesgue measure and an atomic measure supported on rationals.

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How to quantify "shared structure"

1. Furstenberg's Theorem says that non-trivial T_2 and T_3 invariant sets do not have too much shared structure in the most basic sense: they cannot be equal.
2. How can we quantify shared structure in finer/more quantitative ways? The sets we are interested in are fractal: they are uncountable but of zero Lebesgue measure, and have some form of (sub)-self-similarity.
3. Geometry helps quantify common structure. For example, if two sets $A, B \subset \mathbb{R}$ have no shared structure one expects the sumset

$$A + B = \{a + b : a \in A, b \in B\}$$

to be "as large as possible" and the intersection $A \cap B$ and $A \cap (\lambda B + t)$ to be "as small as possible".

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Hausdorff Dimension

- Best exponent for coverings of the set by balls of arbitrary (possibly different) radii:

$$\dim_H(A) = \inf\left\{ s : \inf\left\{ \sum_i r_i^s : A \subset \cup_i B(x_i, r_i) \right\} = 0 \right\}$$

- Gives a notion of "size" for sets in $\mathbb{R}^d$, varies between 0 and d, gives the right size to smooth objects, is invariant under bi-Lipschitz maps, is countably stable, assigns size $\log 2/\log 3$ to the middle-thirds Cantor set,...
- If $A \subset \mathbb{T}$ is T_p-invariant, then $\dim_H A = h_{\text{top}}(A)/\log p$.
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Furstenberg's sumset conjecture

In all conjectures, p, q are rationally independent (not powers of a common integer). E.g. 2 and 3, or 6 and 12 (but not 8 and 16).

Conjecture 1

Let A, B be closed and T_p, T_q invariant. Then

$$\dim_H(A+B) = \max(\dim_H(A) + \dim_H(B), 1).$$

Motivation

- *One "typically" expects the formula above to hold. For example, for arbitrary sets A, B it holds" that*

  $$\dim_H(A + \lambda B) = \max(\dim_H(A) + \dim_H(B), 1) \text{ for almost all } \lambda \in \mathbb{R}$$

- *Moreover, the right-hand side is always a trivial upper bound.*
- *For a strict inequality to occur, A and B must have "shared structure at many scales".*

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Solution to Furstenberg's sumset conjecture

Theorem (Y.Peres-P.S. 2009, F. Nazarov-Y.Peres-P.S. 2012)

If A, B are a p-Cantor set and a q-Cantor set, then

$$\dim_H(A + \lambda B) = \min(\dim_H(A) + \dim_H(B), 1) \text{ for all } \lambda \in \mathbb{R} \setminus \{0\}.$$

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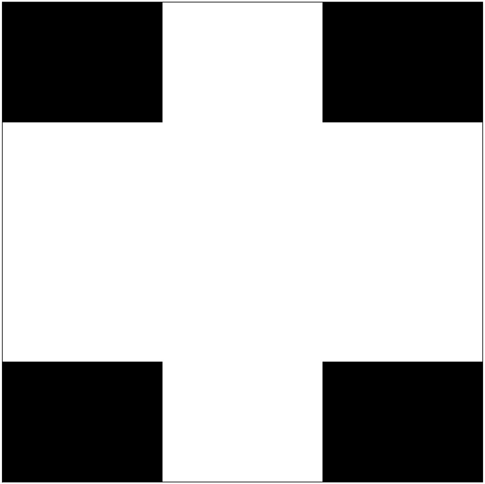
Product, projection, fiber



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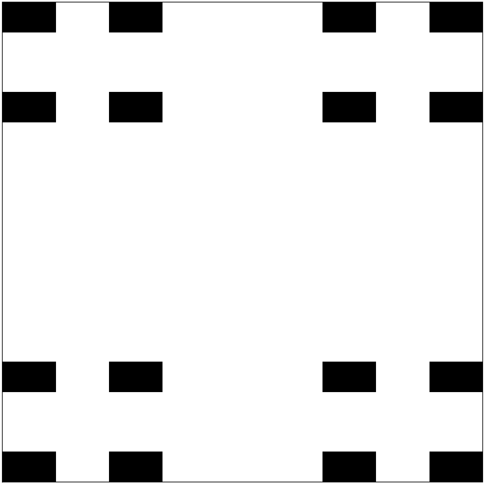

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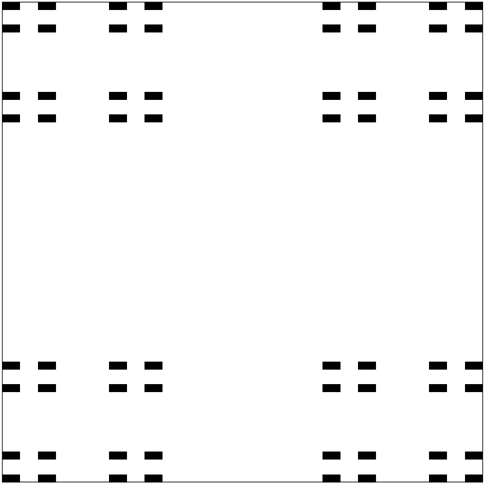

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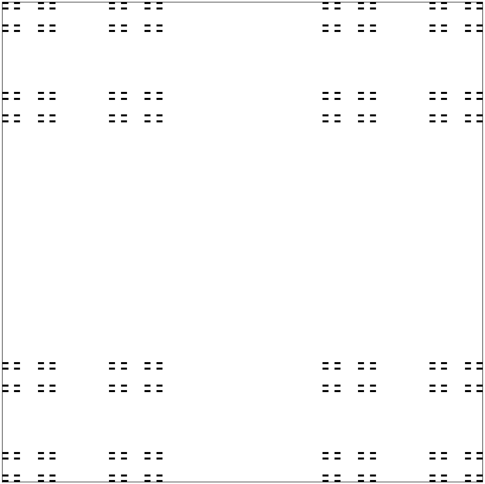

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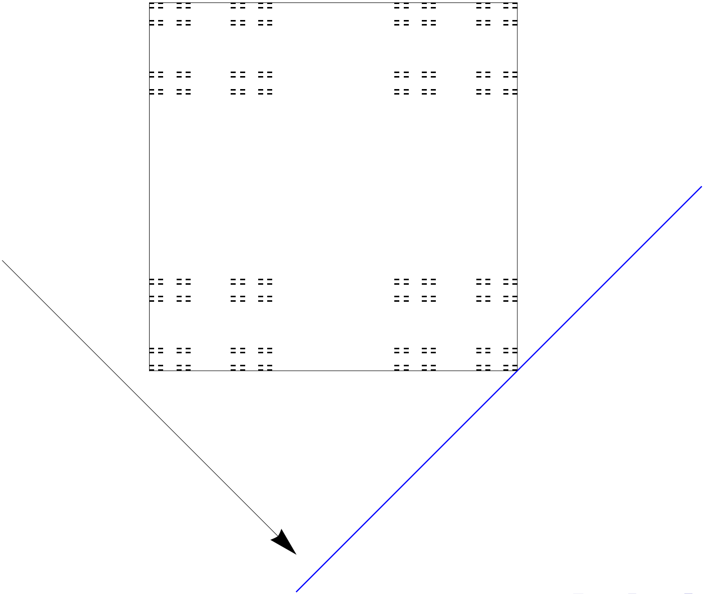

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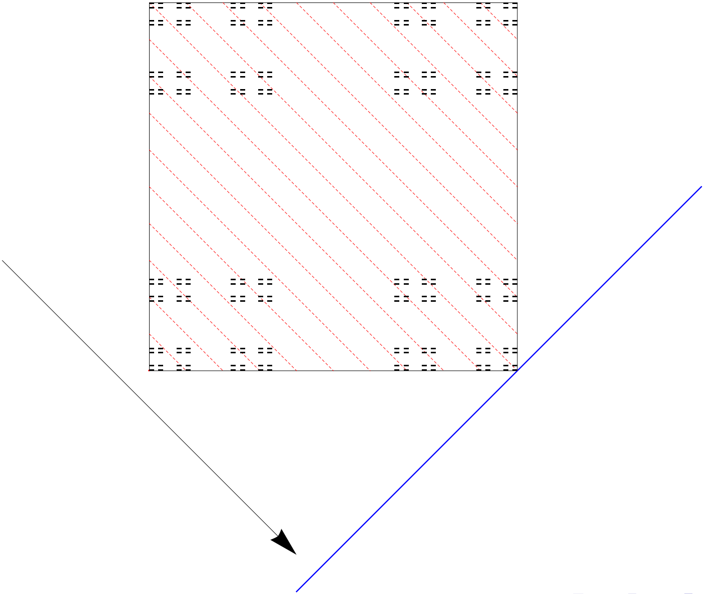

More general notions of shared structure?

- I argued that if

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Furstenberg's intersection conjecture

Conjecture 2 (Furstenberg 1969)

Let A, B be closed and invariant under T_p, T_q (seen as subsets of $\mathbb{R}$). Then for every affine bijection $f : \mathbb{R} \to \mathbb{R}$,

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Motivation

- *It is known that for arbitrary sets A, B one cannot do better than the right-hand side. Counting heuristics show that the RHS is the "average size" of an intersection.*
- *Conjecture 2 is far stronger than Conjecture 1. Heuristically, the sumset $A + B$ is "large" if "many" fibers are "small". The conjecture asserts that all fibers are small.*

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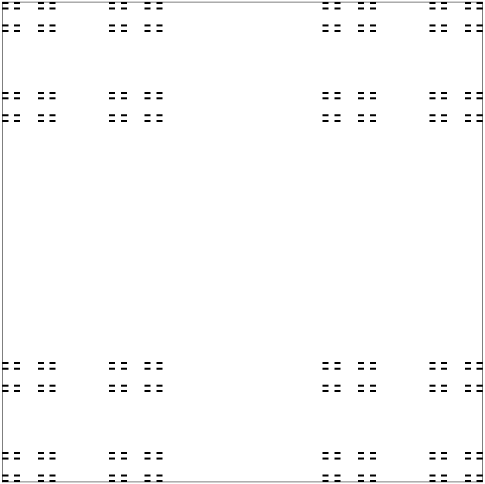

Our old friend again: $A \times B$.

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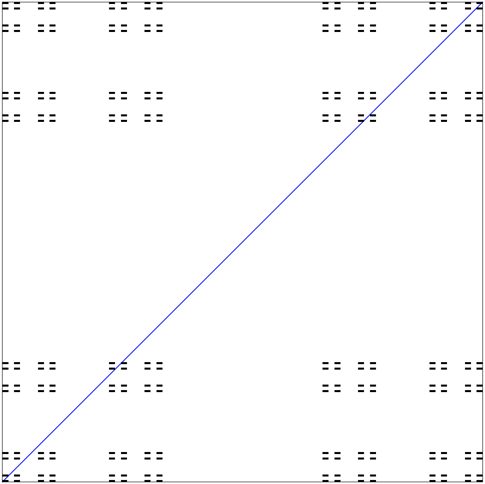

$A \times B \cap$ diagonal $= A \cap B$.

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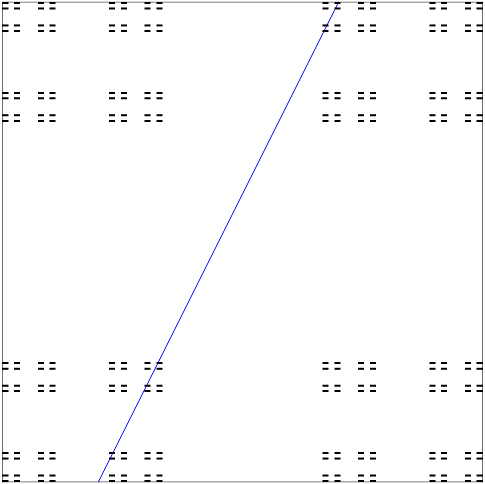

$A \times B \cap$ any line $= A \cap$ affine image of B.

A corollary on subsets of integers

Corollary

Let A be the natural numbers with digits $0, 3$ *in base* 4*, and B the natural numbers with digits* $1, 2, 7$ *in base* 10*. Then*

$$\lim_{n\to\infty} \frac{\log |A \cap B \cap \{1, \ldots, n\}|}{\log n} = 0,$$

in other words, given $\varepsilon > 0$*,*

$$|A \cap B \cap \{1, \ldots, n\}| \leq n^{\varepsilon} \quad \textit{for n large enough.}$$

Question (Related to another conjecture of Furstenberg)

Is $A \cap B$ *finite?*

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Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
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Reduction to a problem in multifractal analysis

- By a standard argument, it is enough to prove the theorem when A, B are a p-Cantor set and a q-Cantor set respectively (with digit sets $D_1 \subset \{0, 1, \ldots, p-1\}, D_2 \subset \{0, 1, \ldots, q-1\}$).
- There are natural measures μ, ν on A, B (Hausdorff measure, measure of maximal entropy, they all agree).
- Let

$$\eta_t = \mu * S_t \nu$$

where $S_t x = tx$ scales by x. Alternatively, η_t is the push-down measure of $\mu \times \nu$ under the linear projection $(x, y) \mapsto x + ty$.
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For all $t \neq 0$,

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The theorem says that η_t is very uniformly distributed in its support $A + tB$ with no points of "larger than expected" mass.

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Multifractal analysis $\to$ intersections II

Proof of Furstenberg's conjecture assuming theorem.

Let

$$s = \dim_H(A) + \dim_H(B) = \dim_H(A \times B).$$

Suppose

$$d = \dim_H(A \cap (tB + u)) > \min(s - 1, 0)$$

Let $u \in I \in \mathcal{D}_n$ with $n \gg 1$. Then, writing $P(x, y) = x + ty$, we have $A \cap (tB + u) \subset P^{-1}(I)$ so that

$$\eta_t(I) = (\mu \times \nu)(P^{-1}(I)) \gtrsim 2^{dn} 2^{-sn}.$$

It follows that

$$2^{\min(s,1)(1-q)n} \geq \sum_{I \in \mathcal{D}_n} \eta_x(I)^q \gtrsim \left(2^{dn} 2^{-sn}\right)^q.$$

This is a contradiction if q is large enough. □

Self-similarity

- $$\mu \sim \sum_{n=1}^{\infty} X_n p^{-n}, \quad \nu \sim \sum_{n=1}^{\infty} Y_n q^{-n}$$
  with X_n, Y_n independent and uniform in D_1, D_2 respectively.
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Theorem (P.S. 2016)

If (η_t) *is a family of "dynamical self-similar measures" where the driving dynamics is uniquely ergodic + some regularity hypotheses, then*

$$D_q(\eta_t) = \textit{what you expect} \quad \textit{for all } t \textit{ and } q > 1.$$

Remark

As corollaries of this theorem, beyond Furstenberg's conjecture I get applications to:

1. *The dimensions and densities of self-similar measures, including Bernoulli convolutions,*
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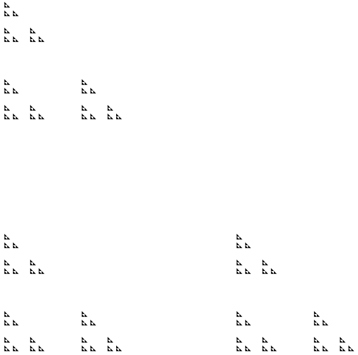

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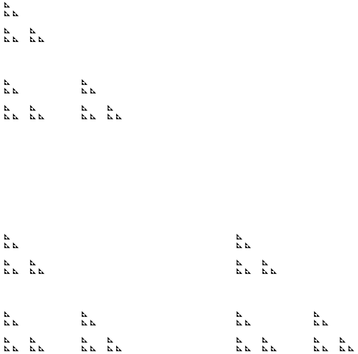

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¡¡¡Muchas gracias!!!