Phase separation phenomenon for one-dimensional long-range Ising models

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The phase separation phenomena belongs to the daily experiences.
The simplest one is when you mix gently a small quantity of oil into water:
Use a plate, put water in it, add softly a spoon of oil in the middle and while mixing try to avoid that the oil touch the plate. Then after some time you find that the small drops of oil will coagulate together to give a large droplet. However looking well you see that there are some very small drops of oil inside the part where the water stand while inside the largest droplet of oil there are also tiny water drops.

To have fun, you can change the ingredients and the temperature.
I recommend to do it with children, teenagers, or during sunday lunch.

- try to add chocolate powder after the large droplet was formed; then mix everything;
- try to add liquid soap to the water (as before);
- change the quality of oil (olive oil on the one hand and multicomponent oil on the other hand);
- start with hot water, put the oil and continue;
- do it in winter/summer (outside).

There is clearly a dynamical aspect in this experiment ("*after some time*") however I will concentrate on the static aspect that is I will try to describe the result of the experiment (*a large droplet of oil inside water*) by using equilibrium statistical mechanics.

However I will not consider the problem of oil and water but a simpler one that occurs also into magnets.

Formalism

Let us introduce $\sigma \equiv (\sigma_i, i \in \mathbb{Z}) \in \{-1,+1\}^{\mathbb{Z}}$ meaning that at each site in $\mathbb{Z}$ we have a variable in $\{-1,+1\}$.

For all finite subsets $\Lambda \subset \mathbb{Z}$ (say an interval) for all configurations $\sigma_\Lambda = (\sigma_i, i \in \Lambda)$ we call Hamiltonian in the finite volume Λ

$$H(\sigma_\Lambda) = -\frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J(i-j)\sigma_i\sigma_j$$

here $J(i-j)$ is a two body interaction $J(i-j) = J(j-i)$ and $\sum_{j\in\mathbb{Z}} |J(i-j)| < \infty$.

If ω is another given configuration in $\{-1,+1\}^{\mathbb{Z}}$ we define the Hamiltonian in the volume Λ with the boundary condition ω by

$$H(\sigma_\Lambda, \omega_{\Lambda^c}) = H(\sigma_\Lambda) + W(\sigma_\Lambda, \omega_{\Lambda^c})$$

with

$$W(\sigma_\Lambda, \omega_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J(i,j)\sigma_i\omega_j.$$

Then consider

$$\mu_\Lambda^\omega(\sigma_\Lambda) = \frac{1}{Z_\Lambda(\omega)} e^{-\beta H(\sigma_\Lambda, \omega_{\Lambda^c})}$$

and interpret it as the conditional density (of σ_Λ) given the configuration ω outside Λ.

A Gibbs measure will be probability measure ν on $\{-1,+1\}^{\mathbb{Z}}$ such that for all intervals $\Lambda \subset \mathbb{Z}$, his conditional expectation given $\mathcal{B}_{\Lambda^c}$ (the σ–algebra generated by $(\omega_i, i \in \Lambda^c)$)) is

$$E_{\nu}[f|\mathcal{B}_{\Lambda^c}](\omega) = \frac{1}{Z_\Lambda(\omega)} \sum_{\sigma_\Lambda} f(\sigma_\Lambda, \omega_{\Lambda^c}) e^{-\beta H(\sigma_\Lambda, \omega_{\Lambda^c})} \quad \nu \ a.s.$$

for all cylindrical function f and

$$Z_\Lambda(\omega) = Z_\Lambda(\beta, \omega) = \sum_{\sigma_\Lambda} e^{-\beta H(\sigma_\Lambda, \omega_{\Lambda^c})}$$

The set of Gibbs measures $\mathcal{G}$ is convex, compact for the narrow topology, the set of extremal points are called Gibbs states and are supposed to describe the different phases of the system.

We consider the case where if $i \neq j$,

$$J(1) = 1 + J$$

where J is considered a parameter. while

$$J(i-j) = \frac{1}{|i-j|^{2-\alpha}} \quad \text{if } |i-j| > 1$$

with $0 \leq \alpha < 1$.
Note that for $\alpha < 0$ we have

- an unique Gibbs state (Ruelle 68, Dobrushin 69) at all β;
- analyticity of the free energy (Dobrushin 73);
- the decay of the two points correlation function is the same as the potential.

However when $\alpha = 0$ we know

- For β large enough there is at least two Gibbs states that are $\mu^+ \neq \mu^-$ (Fröhlich & Spencer (1982));
- At $\beta = \beta_c$ the spontaneous magnetisation is discontinuous, the Thouless effect (Aizenman, Chayes, Chayes & Newman (88);
- there exists an interval of β where the decay of the two points correlation function depends on β, a Berezinski, Kosterlitz & Thouless phase transition (Imbrie & Newman 88).

When $0 < \alpha < 1$ we have

- For β large enough there is at least two Gibbs states this was proved first by Dyson 68 by comparison with the hierarchical model;
- then by Cassandro, Ferrari, Merola & Presutti 05 (for $0 \leq \alpha < \alpha_+ = (log3/log2) - 1$), by a Peierls Argument
- then extended by Littin & Picco 16 to the whole range $0 \leq \alpha < 1$

Note that by a result of Burkov & Sinai (83) all the Gibbs states are translation invariant.

In a recent article Aizenman, Duminil-Copin & Sidoravicius (2015) proved that for $0 < \alpha < 1$ the spontaneous magnetisation is continuous (at $\beta = \beta_c$) when $0 < \alpha < 1$.

Let us come back to μ^+ and μ^-.

There exists a $\beta_c = \beta_c(\alpha)$ such that when $\beta \leq \beta_c$ we have $\mu^+ = \mu^-$ while when $\beta > \beta_c(\alpha)$, we have $\mu^+ \neq \mu^-$.

In particular if $\beta > \beta_c(\alpha)$ then $\mu^+[\sigma_0] = m_\beta > 0$, the so-called spontaneous magnetization while $\mu^-[\sigma_0] = -m_\beta$.

When $\beta > \beta_c(\alpha)$, the typical configurations for μ^+ are informally made of "a sea of $+1$ with islands of -1" .

While the typical configurations for the μ^- are informally made of "a sea of minuses and some island of pluses".

The presence of a mixture in the oil-water experiment above, can be though naively at the level of thermodynamic as the fact that the density of the mixture is in between the density of the water and the density of the oil.

It is expected (but not already proved, we work on it with Jorge Littin) that for all β large enough, (ideally for all $\beta > \beta_c(\alpha)$) all the Gibbs measure are convex combination of the μ^+ and μ^- Gibbs state.
(*i.e.* there are just 2 extremal points in $\mathcal{G}_\beta$.)

This means that for all β large enough (or larger than $\beta_c(\alpha)$) $\forall \mu \in \mathcal{G}$, we should have

$$\mu = (1-\rho)\mu^+ + \rho\mu^-$$

for some $\rho \in [0,1]$.
In particular, for all $i \in \mathbb{Z}$ we should have $\mu[\sigma_i] = (1-2\rho)m_\beta$ where $m_\beta > 0$ is the so called spontaneous magnetisation $m_\beta = \mu^+[\sigma_0]$.
In particular to have $\mu[\sigma_i] = m$ it is enough to take

$$\rho = \rho(m) = \frac{1}{2}\left(1 - \frac{m}{m_\beta}\right)$$

However the phenomena of phase separation has experimentally a spatial behaviour that is clearly not present in such a translation invariant result $\mu[\sigma_i] = (1 - 2\rho)m_\beta \ \forall i \in \mathbb{Z}$

We would like to describe something like :

One of the two phases is attracted by the boundary of the container (plate, glass etc) and the other one is repulsed and forms a droplet that should be centred in the container.

In particular this means that the theory of Gibbs states is not adapted to describe spatially the phenomenon of phase coexistence.
The main question is what can we do to recover a spatial structure.

Here come Minlos & Sinai in 67 and 68 that have the remarkable idea to consider :

Given an $\epsilon_0 > 0$, $m \in]-m_\beta, +m_\beta[$, and β sufficiently large, they consider the constraint

$$\mathcal{A} = \mathcal{A}(m, \epsilon_0) = \left\{ \sigma_\Lambda : \left| \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma_i - m \right| \leq \epsilon_0 m_\beta \right\}$$

that is the empirical mean takes a value m, up to $\epsilon_0 m_\beta$, m being strictly smaller (in modulus) than the mean value m_β.

This is a large deviation from the mean under μ^+, which is m_β and has a μ_Λ^+-probability that goes to zero as $|\Lambda| \uparrow \infty$.

To have an idea of the μ_Λ^+ measure of $\mathcal{A}(m, \epsilon_0)$, in our one-dimensional case we have, when β is large enough

$$\mu_\Lambda^+[\mathcal{A}(m, \epsilon_0)] \approx e^{-\frac{2\beta}{\alpha(1-\alpha)}(\rho(m)|\Lambda|)^\alpha}$$

where again

$$\rho(m) = \frac{1}{2}\left(1 - \frac{m}{m_\beta}\right).$$

The remarkable idea of Minlos & Sinai was to consider the conditional measure

$$\mu_\Lambda^+ \left[\ldots \middle| \mathcal{A}(m, \epsilon_0) \right]$$

and find what are the typical configuration for this conditioned measure when the volume become very large.

This is related to quasi-stationnary measure, a first exemple was given in Moscow around 1947 by Akiva M. Yaglom (which is a student of Kolmogorov) for branching processes.

We are not interested in a large deviations principle but merely to describe what are the typical configurations in a volume Λ, given $\mathcal{A}$ as the volume become very large.

This problem was first considered and solved for the two dimensional Ising model with nearest neighbour interactions by Minlos & Sinai in 1967-68 in one of the most important paper of Mathematical Statistical Mechanics where

"many important ideas, which were later on developed in [Mathematical] Statistical Mechanics were in germs in it".

Heuristics

To have an idea of what could happen let us look at what could be a zero temperature argument. First of all, consider the excess energy with respect to the ground state which is all the spin are $+1$ let us denote it by

$$H^+(\sigma_\Lambda, 1) = \frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J(i,j) 1_{\{\sigma_i \neq \sigma_j\}} + \sum_{i\in\Lambda} \sum_{j\in\Lambda^c} J(i,j) 1_{\{\sigma_i \neq 1\}}.$$

To have an empirical mean of the order m we should have a density of -1 as $(1-m)/2$ and of $+1$ as $(1+m)/2$ so let $I_1^-, \dots, I_n^-$ the intervals (within Λ) where the variables are -1.

with

$$\frac{1}{|\Lambda|} \sum_{i=1}^{n} |I_i^-| = (1-m)/2$$

By using rearrangement argument, the minimum of $H^+(\sigma_\Lambda, 1)$ under this constraint is reached when all these intervals are merged into a single one, say $I^*(\rho)$.

However there is no reason to use a "symmetric" rearrangement here. In particular the $I^*(\rho)$ can be located anywhere within Λ.

With this picture in mind one can hope that adding a very small temperature something of this "picture" could remain valid.
However there are two aspects in the previous picture:

- **I** : there is a large interval with only minuses;
- **II** : it can be located anywhere.

At finite (but large) β, **I** is a little too much, we should accept some fluctuations. In the experiment with oil and water there are the small drops of water inside the large droplet of oil. Moreover one should have some fuzzy zone around the boundary of the interval. So one has to have an idea of what are the "smallest" fluctuations that are called the first excitations.

Concerning the point **II** one makes a scaling,that send $\Lambda \to [-1, +1]$.
The question is : adding a non zero temperature (a disorder) will or will not change this heuristic picture on the location ??

It happens that the zero temperature aspect **I** + small fluctuations is correct (for $0 \leq \alpha < (\log 3/log 2) - 1 = \alpha_+$)

Theorem: *For all* $0 < \alpha < \alpha_+$, *there exists a* $\beta_0(\alpha)$ *such that for all* $\beta > \beta_0(\alpha)$ *with conditional probability going to 1 when* $|\Lambda| \uparrow \infty$, *conditionally with respect to the set* $\mathcal{A}(m, \epsilon_0)$, *(with* $\epsilon_0 = |\Lambda|^{-\alpha/4}$*) the typical configurations are made of an interval of length* $\rho(m)|\Lambda|$ *(that is* $\rho(m)$ *in the macro scale) where*

$$\rho(m) = \frac{1}{2}(1 - \frac{m}{m_\beta})$$

with a fuzzy zone of order $\epsilon_c = |\Lambda|^{-\alpha(1-\alpha)/4}$ *(in the macroscale). There are inside this interval runs of* $+1$ *and* -1, *where the runs of* $+1$ *are all smaller than* $\epsilon_s = |\Lambda|^{-\alpha/4}$ *(in the macro scale), in such a way that the effective macroscopic length of the interval is inside*

$$\rho(m) \pm 6\epsilon_c.$$

For $\alpha = 0$ *the same old just the various* ϵ *have to be chosen as* $(\log |\Lambda|)^{-\gamma}$.

What's about the localisation of the interval ???

Here some unexpected phenomena occurs:

When $0 < \alpha < \alpha_+$*, if* β *is large enough, in the scaling* $\Lambda \to [-1, +1]$*, always conditionally on* $\mathcal{A}(m, \epsilon_0)$*, the localisation of center of the interval goes in distribution to a Dirac measure concentrated at the point* 0.

That is adding the disorder due to the temperature the system chooses the centred situation with no fluctuations in the macro scale.
While

When $\alpha = 0$*, if* β *is large enough, under the same scaling, the center of the interval converges in distribution to a random variable with an explicit density absolutely continuous with respect to the Lebesgue measure.*

When $\beta \uparrow \infty$ one recover an uniform distribution.