Pointwise convergence of some averages for commuting transformations. (joint work with Wenbo Sun)

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A fundamental question in ergodic theory is the convergence (in some sense) of the average

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x)\cdots f_d(T_d^n x).$$

Here $(X, \mathcal{X}, \mu)$ is a probability space and $T_i \colon X \to X$ are measurable, measure-preserving $(\mu(T_i^{-1}A) = \mu(A),\ A \in \mathcal{X})$.

Motivated by Furstenberg's proof (1977) of Szemerédi Theorem.

Linked to additive combinatorics and number theory.

Some historical results:

L^2 convergence:

One single transformation ($T_i = T^i$) Furstenberg (1977): $d = 2$
Conze and Lesigne (1988); Furstenberg and Weiss (1996);
Host and Kra (2001): $d = 3$.
Host and Kra (2005); Ziegler (2007): $d \in \mathbb{N}$.

Commutative case
Tao (2007, finitary), Towsner (2008, non-standard analysis), Austin (2009, ergodic), Host (2009, ergodic: using “cubes”).

Nilpotent case Walsh (2012, finitary).

Problems of pointwise and L^2 convergence are really different.

Pointwise convergence:

Pointwise convergence results

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Pointwise convergence:

Bourgain (1990) :

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x).$$

Huang, Shao and Ye (2014) : distal case.

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{n}x)f_2(T^{2n}x)\cdots f_d(T^{dn}x).$$

Proof using very special (in the sense of particular properties of some structures) topological models.

D. and Sun (2015) : T_1, T_2 commuting distal

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x)$$

Back to the L^2 setting

Expression for the limit? Only some results for a single transformation (X, μ, T)

Von Neumann Ergodic Theorem

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \mathbb{E}(f \mid \mathcal{I}(T))(x) = \int f \, d\mu_x$$

where $\mu = \int \mu_x$ is the disintegration of μ over $\mathcal{I}(T)$. The measure μ_x is ergodic for μ-a.e $x \in X$.

Nilsystems as characteristic factors help to have an expression for the limit.
Sketch:
Host and Kra (2005) There exists a factor $\mathcal{Z}_{d-1}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x)$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_1(T^n x) g_2(T^{2n} x) \cdots g_d(T^{dn} x)$$

where $g_i = \mathbb{E}(f_i \mid \mathcal{Z}_{d-1})$.

Structure theorem: The factor $\mathcal{Z}_{d-1}$ is an inverse limit of $(d-1)$-step nilmanifolds.

This reduces to study the average when (X, μ, T) is a nilmanifold.
In this case

$$N_x = \overline{\{(T \times T^2 \times \cdots \times T^d)(x, \ldots x)\}}$$

is also a nilmanifold (Leibman (1998)).

The limit is well studied:

Lesigne (1989) $d = 3$;
Ziegler (2002) all d;
Bergelson, Host and Kra (2005) all d.

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x) \to \int f_1 \otimes \cdots \otimes f_d \, d\mu_x$$

where μ_x is ergodic for $T \times \cdots \times T^d$ and is the Haar measure of a nilsystem.

How about the commuting case?

No structures theorems are know and there is no expression for the limit.

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Theorem (D. and Sun, 2016)

Let $(X, \mu, T_1, \ldots, T_d)$ be a m.p.s with commuting transformations and $f_1 \ldots, f_d \in L^\infty(\mu)$. Then, in L^2

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x) \cdots f_d(T_d^n x) \to \int f_1 \otimes \cdots \otimes f_d \, d\mu_x$$

where μ_x is ergodic for $T_1 \times \cdots \times T_d$.

We do not know if these measures are related to nilsystems. Probably yes.

Combined with methods by Huang, Shao and Ye.

Theorem (D. and Sun, 2016)

Let $(X, \mu, T_1, \ldots, T_d)$ be ergodic and distal for $\langle T_1, \ldots, T_d \rangle$ and $f_1 \ldots, f_d \in L^\infty(\mu)$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x) \cdots f_d(T_d^n x)$$

converges μ-a.e $x \in X$. The limit of course is $\int f_1 \otimes \cdots \otimes f_d \, d\mu_x$

Proof ideas.

Recall some definitions

A measure preserving system $(X, \mathcal{X}, \mu, T_1, \ldots, T_d)$ is a probability space $(X, \mathcal{X}, \mu)$ and $T_i \colon X \to X$ measurable, measure-preserving $(\mu(T_i^{-1}A) = \mu(A),\ A \in \mathcal{X})$, $i = 1, \ldots, d$.

$(X, \mathcal{X}, \mu, T)$ is ergodic if $A = T_i^{-1}A$ for all i implies $\mu(A) = 0$ or 1.

A factor map π is a measure preserving transformation from (X, μ) to (Y, ν) such the diagram below commutes

$$\begin{array}{ccc} (X,\mu) & \xrightarrow{\;T_i\;} & (X,\mu) \\ \pi \downarrow & & \downarrow \pi \\ (Y,\nu) & \xrightarrow[\;T_i\;]{} & (Y,\nu) \end{array}$$

$\pi\mu = \nu$ and $\pi \circ T_i = T_i \circ \pi$, $i = 1, \ldots, d$.

Some less classical definitions: Let $(X, \mu, T_1, \ldots, T_d)$ a m.p.s.
Define inductively the *Host's measures*

$$\mu_{T_1} = \mu \times_{\mathcal{I}(T_1)} \mu \quad \text{measure on } X^2.$$

i.e.

$$\int_{X^2} f_1 \otimes f_2 d\mu_{T_1} = \int_X \mathbb{E}(f_1 \mid \mathcal{I}(T_1))\mathbb{E}(f_1 \mid \mathcal{I}(T_1)) d\mu$$

$$\mu_{T_1,T_2} = \mu_{T_1} \times_{\mathcal{I}(T_2 \times T_2)} \mu_{T_1} \quad \text{measure on } X^4$$

$$\mu_{T_1,T_2,T_3} = \mu_{T_1,T_2} \times_{\mathcal{I}(T_3^4)} \mu_{T_1,T_2} \quad \text{measure on } X^8$$

and so on.

They are useful because allow us to build seminorms that control the multiple averages.
For $f \in L^\infty(\mu)$, define

$$|||f|||_{\mu,T_1,\ldots,T_d} = \left(\int f \otimes \cdots \otimes f \, d\mu_{T_1,\ldots,T_d} \right)^{1/2^d}$$

It is a seminorm and does not depend on the order chosen for $T_1, \ldots, T_d$.

Remains the same if we change T_i by T_i^{-1}.

Theorem (Host)

Let $(X, \mu, T_1, \ldots, T_d)$ *a measure preserving system. Then*

$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^n x) \cdots f_d(T_d^n x) \right\|_{L^2(\mu)}$$

$$\leq |||f_1|||_{\mu, T_1, T_1^{-1}T_2, \ldots, T_1^{-1}T_d}$$

$$|||f_2|||_{\mu, T_2, T_2^{-1}T_1, \ldots, T_2^{-1}T_d}$$

$$\vdots$$

$$|||f_d|||_{\mu, T_d, T_d^{-1}T_1, \ldots, T_d^{-1}T_{d-1}}$$

Similar result holds for self joinings.

Recall a proof for the L^2-convergence (originally due to Austin (2009)) Use induction in d.

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x)\cdots f_d(T_d^n x)$$

Suppose f_1 is measurable with respect to

$$\mathcal{I}(T_1) \vee \bigvee_{i\geq 2} \mathcal{I}(T_1^{-1}T_j).$$

Say

$$f_1 = g_1 \cdot g_2 \cdots g_d$$

where g_1 is $\mathcal{I}(T_1)$-measurable and g_i is $\mathcal{I}(T_1^{-1}T_i)$-measurable. Then

$$f_1(T_1^n x) = g_1(x) \cdot g_2(T_2^n x)\cdots g_d(T_d^n x)$$

Introducing this in the average we have

$$g_1(x)\frac{1}{N}\sum_{n=0}^{N-1}(f_2\cdot g_2)(T_2^n x)\cdots(f_d\cdot g_d)(T_d^n x)$$

Use induction hypothesis.

It remains to study what happens when

$$\mathbb{E}(f_1\mid \mathcal{I}(T_1)\vee\bigvee_{i\geq 2}\mathcal{I}(T_1^{-1}T_i))=0$$

Motivated by Austin's work, Host introduced special extensions. $(X, \mu, T_1, \ldots, T_d)$ is magic for $T_1, T_1^{-1}T_2, \ldots, T_1^{-1}T_d$ if

$$\mathbb{E}(f \mid \mathcal{I}(T_1) \vee \bigvee_{i\geq 2} \mathcal{I}(T_1^{-1}T_i)) = 0 \quad \text{if and only if} \quad |||f|||_{\mu, T_1, T_1^{-1}T_2 \ldots, T_1^{-1}T_d} = 0$$

In particular, in a magic system, if $\mathbb{E}(f_1 \mid \mathcal{I}(T_1) \vee \bigvee_{i\geq 2} \mathcal{I}(T_1^{-1}T_i)) = 0$ then the average goes to 0.

Theorem (Host (2009))

Every system $(Y, \nu, T_1, \ldots, T_d)$ admits an extension $(X, \mu, T_1, \ldots, T_d)$ which is magic for $T_1, T_1^{-1}T_2, \ldots, T_1^{-1}T_d$.

From now, we focus on $d = 2$

Let (X, μ, S_1, S_2, S_3) be a system with three transformations.

Write $R_{1,2} = S_1^{-1} S_2$, $R_{1,3} = S_1^{-1} S_3$ and $R_{2,3} = S_2^{-1} S_3$.

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The system (X, μ, S_1, S_2, S_3) is *mystical* if it is magic for:

-$R_{1,2}, R_{2,3}$

-$R_{1,3}, R_{2,3}$.

Proposition

Every system admits a mystical extension.

The Furstenberg-Ryzhikov self-joining of (S_1, S_2, S_3) is

$$\mu^F(f_1 \otimes f_2 \otimes f_3) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f_1(S_1^n x) f_2(S_2^n x) f_3(S_3^n x) d\mu(x)$$

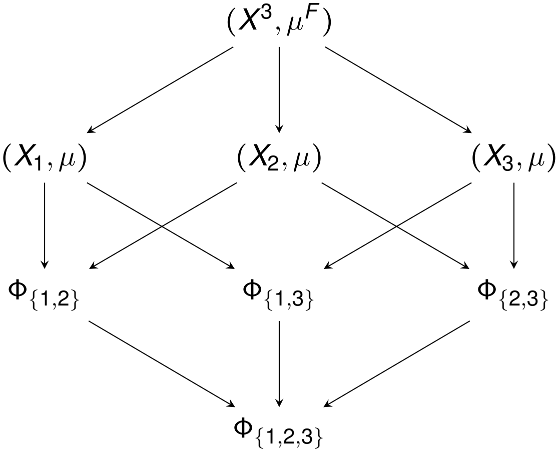
The joining μ^F is invariant under

$$S_1 \times S_1 \times S_1,$$
$$S_2 \times S_2 \times S_2,$$
$$S_3 \times S_3 \times S_3,$$
$$S_1 \times S_2 \times S_3.$$

Some notation: For $J \subseteq \{1, 2, 3\}$, let Φ_J denote the σ-algebra (of X) invariant under all transformations $S_i^{-1} S_j$ $i, j \in J$. For instance

$$\Phi_{\{1,2\}} = \mathcal{I}(R_{1,2})$$

Austin's diagram for the Furstenberg-Ryzhikov self joining (for (S_1, S_2, S_3)).



Denote ν^F the projection of μ^F onto the last two coordinates.

Question: study $\mathcal{I}(R_{1,2} \times R_{1,3})$ in (X^2, ν^F).

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$$
\begin{aligned}
&\left\|\mathbb{E}\big((f_2 - \mathbb{E}(f_2 \mid \Phi_{\{1,2\}} \vee \Phi_{\{2,3\}})\big) \otimes f_3 \mid \mathcal{I}(R_{1,2} \times R_{1,3})\right\|_{L^2(\nu^F)} \\
&= \lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \big(f_2 - \mathbb{E}(f_2 \mid \Phi_{\{1,2\}} \vee \Phi_{\{2,3\}})\big) \otimes f_3 \circ (R_{1,2}^n \times R_{1,3}^n) \right\|_{L^2(\nu^F)} \\
&\leq \left|\!\left|\!\left| f_2 - \mathbb{E}(f_2 \mid \Phi_{\{1,2\}} \vee \Phi_{\{2,3\}}) \right|\!\right|\!\right|_{\mu, R_{1,2}, R_{1,2}^{-1} R_{1,3}} \\
&= \left|\!\left|\!\left| f_2 - \mathbb{E}(f_2 \mid \Phi_{\{1,2\}} \vee \Phi_{\{2,3\}}) \right|\!\right|\!\right|_{\mu, R_{1,2}, R_{2,3}}
\end{aligned}
$$

This is 0 if (X, μ, S_1, S_2, S_3) is mystical.

We can replace functions by conditional expectations.

$$\mathbb{E}(f_2 \otimes f_3 \mid \mathcal{I}(R_{1,2} \times R_{1,3}))$$

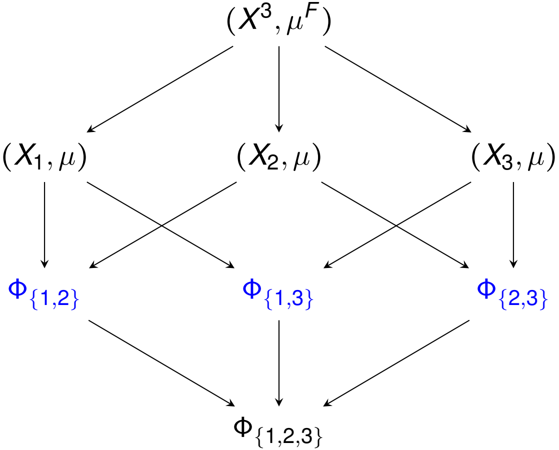
$$=$$

$$\mathbb{E}(\mathbb{E}\left(f_2 \mid \Phi_{\{1,2\}} \vee \Phi_{\{2,3\}}\right) \otimes \mathbb{E}\left(f_3 \mid \Phi_{\{1,3\}} \vee \Phi_{\{2,3\}}\right) \mid \mathcal{I}(R_{1,2} \times R_{1,3}))$$

The last expression is measurable (in (X^3, μ^F)) with respect to

$$\Phi_{\{1,2\}} \vee \Phi_{\{1,3\}} \vee \Phi_{\{2,3\}}$$

$\mathcal{I}(T_2 \times T_3)$ is then measurable with respect to



We can study now $\mathcal{I}(R_{1,2} \times R_{1,3})$ in $\Phi_{\{1,2\}} \vee \Phi_{\{1,3\}} \vee \Phi_{\{2,3\}}$.

Note that $R_{1,2}$ acts trivially in $\Phi_{\{1,2\}}$ and $R_{1,3}$ acts trivially in $\Phi_{\{1,3\}}$.

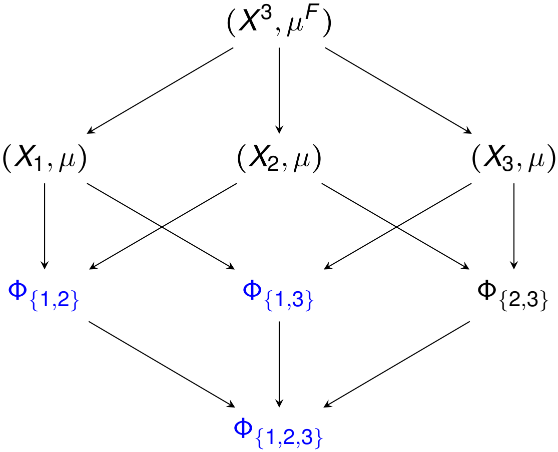
So $\mathcal{I}(R_{1,2} \times R_{1,3})$ is measurable with respect to

$$\Phi_{\{1,2\}} \vee \Phi_{\{1,3\}} \vee \mathcal{I}(R_{1,2})(\Phi_{\{2,3\}})$$

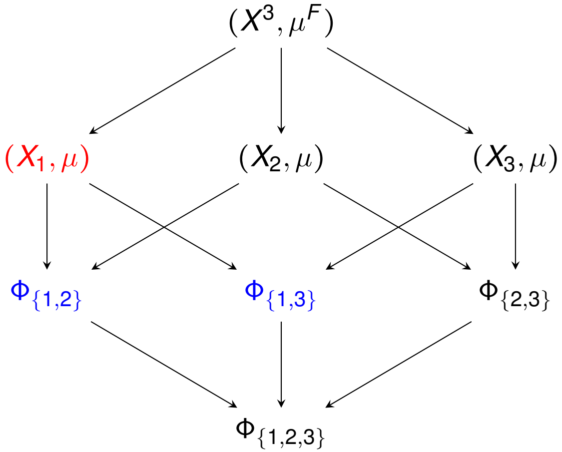
which is nothing but

$$\Phi_{\{1,2\}} \vee \Phi_{\{1,3\}} \vee \Phi_{\{1,2,3\}}.$$

$\mathcal{I}(T_2 \times T_3)$ is then measurable with respect to



$\mathcal{I}(T_2 \times T_3)$ is then measurable with respect to (the first coordinate!



This fact allows to get an expression for the Furstenberg self-joining

$$\mu^F = \int_X \delta_x \times \nu_x^F d\mu(x)$$

where ν_x^F is $R_{1,2} \times R_{1,3}$-ergodic.

Let $F(x)$ the L^2-limit of

$$\frac{1}{N}\sum_{n=0}^{N-1} f_2(R_{1,2}^n x) f_3(R_{1,3}^n x)$$

For $f \in L^\infty(\mu)$,

$$\begin{aligned}
\int_X f(x)F(x) &= \lim \frac{1}{N}\sum_{n=0}^{N-1}\int f(x)f_2(R_{1,2}^n x)f_3(R_{1,3}^n x)d\mu(x) \\
&= \lim \frac{1}{N}\sum_{n=0}^{N-1}\int f(S_1^n x)f_2(S_2^n x)f_3(S_3^n x)d\mu(x) \\
&= \mu^F(f\otimes f_2\otimes f_3) \\
&= \int f(x)\left(\int f_2\otimes f_3 d\nu_x^F\right)d\mu(x)
\end{aligned}$$

where the last equality comes from $\mu^F = \int_X \delta_x \times \nu_x^F d\mu(x)$.
So

$$F(x) = \int f_2 \otimes f_3 d\nu_x^F$$

Arbitrary case

Let (Y, ν, T_2, T_3) be a system with two commuting transformations.

Consider it as a $\mathbb{Z}^3$ dynamical system.

$$(Y, \nu, T_1 = \mathrm{id}, T_2, T_3)$$

Take a mystical extension

$$(X, \mu, S_1, S_2, S_3) \to (Y, \nu, \mathrm{id}, T_2, T_3)$$

Remark that $R_{1,2}$ and $R_{1,3}$ are mapped to T_2 and T_3.

So, it suffices to study

$$\frac{1}{N}\sum_{n=0}^{N-1} f_2(R_{1,2}^n x) f_3(R_{1,3}^n x)$$

in (X, μ, S_1, S_2, S_3) and project it.

Open questions

- Expression for the L^2-limit

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x)\cdots f_d(T_d^n x)$$

when $T_1, \ldots, T_d$ generate a nilpotent group?

- Pointwise convergence under weak mixing assumptions?

- Pointwise convergence under polynomial iterates?

Thanks for your attention!