Large-time behavior of viscous Hamilton-Jacobi equations

A. Quaas

Departamento de Matemática
UTFSM

UNIVERSIDAD TECNICA
FEDERICO SANTA MARIA
Departamento de Matemática

Joint work with G. Barles and A. Rodriguez

Rio.-24-June-2024

The viscous Hamilton-Jacobi equation

Our aim is to study the existence, uniqueness and large-time behavior (LTB) of solutions of

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- The question of LTB is to determine in some sense the behavior of $u(\cdot,t)$ as $t \to +\infty$.

The viscous Hamilton-Jacobi equation

Our aim is to study the existence, uniqueness and large-time behavior (LTB) of solutions of

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- The question of LTB is to determine in some sense the behavior of $u(\cdot, t)$ as $t \to +\infty$.
- The case $m > 2$ in [Barles et al., 2020] (joint work with G. Barles and A. Rodríguez).

The viscous Hamilton-Jacobi equation

Our aim is to study the existence, uniqueness and large-time behavior (LTB) of solutions of

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- The question of LTB is to determine in some sense the behavior of $u(\cdot, t)$ as $t \to +\infty$.
- The case $m > 2$ in [Barles et al., 2020] (joint work with G. Barles and A. Rodríguez).
- The case $1 < m \leq 2$, joint work with A. Rodríguez.

The viscous Hamilton-Jacobi equation

Our aim is to study the existence, uniqueness and large-time behavior (LTB) of solutions of

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- The question of LTB is to determine in some sense the behavior of $u(\cdot, t)$ as $t \to +\infty$.
- The case $m > 2$ in [Barles et al., 2020] (joint work with G. Barles and A. Rodríguez).
- The case $1 < m \leq 2$, joint work with A. Rodríguez.
- The behavior of f and u_0 as $|x| \to +\infty$ is crucial, especially in comparison with previous results.

The viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- For $m = 2$, this is a deterministic version of the Kardar-Parisi-Zhang (KPZ) equation ([Kardar et al., 1986]),

The viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- For $m = 2$, this is a deterministic version of the Kardar-Parisi-Zhang (KPZ) equation ([Kardar et al., 1986]),
- For $m > 1$, (VHJ) is a simple model of superlinear gradient dependence,

The viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u + |Du|^m = f(x) & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{VHJ}$$

- For $m = 2$, this is a deterministic version of the Kardar-Parisi-Zhang (KPZ) equation ([Kardar et al., 1986]),
- For $m > 1$, (VHJ) is a simple model of superlinear gradient dependence,
- (Lasry-Lions 1989 stationary case). Solutions are value functions for stochastic optimal control problems:

$$u(x,t) = \inf_{(a_s)_s} E_x \left[\int_0^t (m-1) m^{-\frac{m}{m-1}} |a_s|^{\frac{m}{m-1}} + f(X_s)\, ds + u_0(X_t) \right]$$

where

$$dX_t = a_t\, dt + dB_t \quad \text{for } t > 0,\ X_0 = x \in \mathbb{R}^N.$$

Prior results

- For bounded domain: [Tabet Tchamba, 2010] ($m > 2$), Barles, Porretta, Tchamba [Barles et al., 2010] ($1 < m \leq 2$).

Prior results

- For bounded domain: [Tabet Tchamba, 2010] ($m > 2$), Barles, Porretta, Tchamba [Barles et al., 2010] ($1 < m \leq 2$).
- Over $\mathbb{R}^N$, overlap of results with [Ichihara, 2012] LTB; Arapostathis, Biswas, Caffarelli [Arapostathis et al., 2019] uniqueness of the Ergodic problem, therein obtained by a combination of PDE, probabilistic and dynamical systems techniques.

Prior results

- For bounded domain: [Tabet Tchamba, 2010] ($m > 2$), Barles, Porretta, Tchamba [Barles et al., 2010] ($1 < m \leq 2$).
- Over $\mathbb{R}^N$, overlap of results with [Ichihara, 2012] LTB; Arapostathis, Biswas, Caffarelli [Arapostathis et al., 2019] uniqueness of the Ergodic problem, therein obtained by a combination of PDE, probabilistic and dynamical systems techniques.
- $\mathbb{R}^N$ Barles, G. and Meireles [2016]. uniqueness of the Ergodic problem.

Prior results

- For bounded domain: [Tabet Tchamba, 2010] ($m > 2$), Barles, Porretta, Tchamba [Barles et al., 2010] ($1 < m \leq 2$).
- Over $\mathbb{R}^N$, overlap of results with [Ichihara, 2012] LTB; Arapostathis, Biswas, Caffarelli [Arapostathis et al., 2019] uniqueness of the Ergodic problem, therein obtained by a combination of PDE, probabilistic and dynamical systems techniques.
- $\mathbb{R}^N$ Barles, G. and Meireles [2016]. uniqueness of the Ergodic problem.
- In the context of weak solutions: [Benachour et al., 2004], [Biler et al., 2004], [Gallay and Laurençot, 2007], [Iagar et al., 2010], [Laurençot, 2009].

Prior results

- For bounded domain: [Tabet Tchamba, 2010] ($m > 2$), Barles, Porretta, Tchamba [Barles et al., 2010] ($1 < m \leq 2$).
- Over $\mathbb{R}^N$, overlap of results with [Ichihara, 2012] LTB; Arapostathis, Biswas, Caffarelli [Arapostathis et al., 2019] uniqueness of the Ergodic problem, therein obtained by a combination of PDE, probabilistic and dynamical systems techniques.
- $\mathbb{R}^N$ Barles, G. and Meireles [2016]. uniqueness of the Ergodic problem.
- In the context of weak solutions: [Benachour et al., 2004], [Biler et al., 2004], [Gallay and Laurençot, 2007], [Iagar et al., 2010], [Laurençot, 2009].
- First-order equations, results employing dynamical systems/optimal control arguments: [Namah and Roquejoffre, 1999], [Barles and Souganidis, 2001], [Barles and Roquejoffre, 2006], [Ichihara and Ishii, 2009], [Ishii, 2008], [Ishii, 2009] (review paper), [Barles et al., 2017].

Bounded domain—well-posedness

For $\Omega \subset \mathbb{R}^N$ an open, bounded set and $g \in C(\partial\Omega)$ consider

$$\begin{aligned} u_t - \Delta u + |Du|^m &= f(x) \quad \text{in } \Omega \times (0, +\infty) \\ u &= g \quad \text{on } \partial\Omega \times (0, +\infty) \\ u(\cdot, 0) &= 0 \quad \text{in } \overline{\Omega}. \end{aligned} \tag{VHJ$_b$}$$

- Local existence in $C^{2,1}(\Omega \times (0, T))$ for some $T \in (0, +\infty]$, [Friedman, 2013], [Quittner and Souplet, 2007].

Bounded domain—well-posedness

For $\Omega \subset \mathbb{R}^N$ an open, bounded set and $g \in C(\partial\Omega)$ consider

$$\begin{aligned} u_t - \Delta u + |Du|^m &= f(x) && \text{in } \Omega \times (0, +\infty) \\ u &= g && \text{on } \partial\Omega \times (0, +\infty) \\ u(\cdot, 0) &= 0 && \text{in } \overline{\Omega}. \end{aligned} \qquad (\mathrm{VHJ}_b)$$

- Local existence in $C^{2,1}(\Omega \times (0, T))$ for some $T \in (0, +\infty]$, [Friedman, 2013], [Quittner and Souplet, 2007].
- Well-posed by [Barles and Da Lio, 2004], assuming *generalized boundary conditions* for $m > 2$ and in the classical sense, for regular data, for $1 < m \leq 2$.

Bounded domain—well-posedness

For $\Omega \subset \mathbb{R}^N$ an open, bounded set and $g \in C(\partial\Omega)$ consider

$$\begin{aligned} u_t - \Delta u + |Du|^m &= f(x) \quad \text{in } \Omega \times (0, +\infty) \\ u &= g \quad \text{on } \partial\Omega \times (0, +\infty) \\ u(\cdot, 0) &= 0 \quad \text{in } \overline{\Omega}. \end{aligned} \tag{VHJ$_b$}$$

- Local existence in $C^{2,1}(\Omega \times (0, T))$ for some $T \in (0, +\infty]$, [Friedman, 2013], [Quittner and Souplet, 2007].
- Well-posed by [Barles and Da Lio, 2004], assuming *generalized boundary conditions* for $m > 2$ and in the classical sense, for regular data, for $1 < m \leq 2$.
- Loss of boundary condition for $m > 2$ many other properties Porretta, Souplet [2017] and [2020], see also Q.;L Rodriguez [2018] .

Bounded domain—the stationary problem

$$\begin{cases} -\Delta w + |Dw|^m = f(x) & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases} \tag{S}$$

- Depending on the data, (S) might not have a solution ([Souplet and Zhang, 2006], [Grenon et al., 2013])

Bounded domain—the ergodic problem

LTB is determined by

$$\begin{cases} \lambda - \Delta\phi + |D\phi|^m = f(x) & \text{in } \Omega, \\ \lambda - \Delta\phi + |D\phi|^m \geq f(x) & \text{on } \partial\Omega \end{cases} \tag{E$_b$}$$

where $\lambda \in \mathbb{R}$ is an unknown, together with $\phi \in C(\Omega)$.

Bounded domain—the ergodic problem

LTB is determined by

$$\begin{cases} \lambda - \Delta\phi + |D\phi|^m = f(x) & \text{in } \Omega, \\ \lambda - \Delta\phi + |D\phi|^m \geq f(x) & \text{on } \partial\Omega \end{cases} \tag{E$_b$}$$

where $\lambda \in \mathbb{R}$ is an unknown, together with $\phi \in C(\Omega)$.

- There exists a unique value $\lambda^* \in \mathbb{R}$ for which there exists a solution $\phi \in C(\Omega)$ (unique up to an additive constant).

Bounded domain—the ergodic problem

LTB is determined by

$$\begin{cases} \lambda - \Delta\phi + |D\phi|^m = f(x) & \text{in } \Omega, \\ \lambda - \Delta\phi + |D\phi|^m \geq f(x) & \text{on } \partial\Omega \end{cases} \tag{E$_b$}$$

where $\lambda \in \mathbb{R}$ is an unknown, together with $\phi \in C(\Omega)$.

- There exists a unique value $\lambda^* \in \mathbb{R}$ for which there exists a solution $\phi \in C(\Omega)$ (unique up to an additive constant).
- Characterization of λ^*:

$$\lambda^* = \sup\{\lambda \in \mathbb{R}^N \mid \exists\psi \in C(\Omega),\ \lambda - \Delta\psi + |D\psi|^m \leq f(x)\}$$

Bounded domain—the ergodic problem

LTB is determined by

$$\begin{cases} \lambda - \Delta\phi + |D\phi|^m = f(x) & \text{in } \Omega, \\ \lambda - \Delta\phi + |D\phi|^m \geq f(x) & \text{on } \partial\Omega \end{cases} \tag{E$_b$}$$

where $\lambda \in \mathbb{R}$ is an unknown, together with $\phi \in C(\Omega)$.

- There exists a unique value $\lambda^* \in \mathbb{R}$ for which there exists a solution $\phi \in C(\Omega)$ (unique up to an additive constant).
- Characterization of λ^*:

$$\lambda^* = \sup\{\lambda \in \mathbb{R}^N \mid \exists \psi \in C(\Omega),\ \lambda - \Delta\psi + |D\psi|^m \leq f(x)\}$$

- Studied in the classical work [Lasry and Lions, 1989]; see further properties in, cf. [Tabet Tchamba, 2010], [Barles et al., 2010].

Bounded domain—the ergodic problem

The *“state constraints”* boundary condition

$$\lambda - \Delta\phi + |D\phi|^m \geq f(x) \quad \text{on } \partial\Omega,$$

is meant in a generalized sense.

Bounded domain—the ergodic problem

The *"state constraints"* boundary condition

$$\lambda - \Delta\phi + |D\phi|^m \geq f(x) \quad \text{on } \partial\Omega,$$

is meant in a generalized sense.

- for $m > 2$, $\phi \in C(\overline{\Omega})$ (i.e., remains bounded),

Bounded domain—the ergodic problem

The *"state constraints"* boundary condition

$$\lambda - \Delta\phi + |D\phi|^m \geq f(x) \quad \text{on } \partial\Omega,$$

is meant in a generalized sense.

- for $m > 2$, $\phi \in C(\overline{\Omega})$ (i.e., remains bounded),
- for $1 < m \leq 2$, $\phi \to +\infty$ as $x \to \partial\Omega$.

Bounded domain—ergodic LTB

- When $m > 2$, if (S) has a solution w, then $u(\cdot, t) \to w$ as $t \to +\infty$, uniformly over $\bar{\Omega}$.

Bounded domain—ergodic LTB

- When $m > 2$, if (S) has a solution w, then $u(\cdot, t) \to w$ as $t \to +\infty$, uniformly over $\bar{\Omega}$.
- If (S) is not solvable, then $\lambda^* < 0$ in (E_b) and $u(\cdot, t) - \lambda^* t \to \phi + c$ as $t \to +\infty$ uniformly over $\bar{\Omega}$, for some $c \in \mathbb{R}$. [Tabet Tchamba, 2010]

Bounded domain—ergodic LTB

- When $m > 2$, if (S) has a solution w, then $u(\cdot, t) \to w$ as $t \to +\infty$, uniformly over $\bar{\Omega}$.
- If (S) is not solvable, then $\lambda^* < 0$ in (E_b) and $u(\cdot, t) - \lambda^* t \to \phi + c$ as $t \to +\infty$ uniformly over $\bar{\Omega}$, for some $c \in \mathbb{R}$. [Tabet Tchamba, 2010]
- For $1 < m \leq 2$, LTB also depends on m: e.g., if $1 < m \leq \frac{3}{2}$ and $\lambda^* < 0$, then a partial convergence holds: $\frac{u(x,t)}{t} \to \lambda^*$ locally uniformly in Ω but it can happen that $u(\cdot, t) - \lambda^* t \to -\infty$ in Ω [Barles et al., 2010].

Unbounded domains—$m > 2$

Theorem (Barles, R., Quaas, 2020)

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $u_0 \in C(\mathbb{R}^N)$ are bounded from below. Then, there exists a unique, nonnegative, continuous solution of (VHJ).

Unbounded domains—$m > 2$

Theorem (Barles, R., Quaas, 2020)

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $u_0 \in C(\mathbb{R}^N)$ are bounded from below. Then, there exists a unique, nonnegative, continuous solution of (VHJ)*.*

- *Existence:* aproximation by solutions on bounded domains, compactness arguments.

Unbounded domains—$m > 2$

Theorem (Barles, R., Quaas, 2020)

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $u_0 \in C(\mathbb{R}^N)$ are bounded from below. Then, there exists a unique, nonnegative, continuous solution of (VHJ).

- *Existence:* aproximation by solutions on bounded domains, compactness arguments.
- *Uniqueness:* comparison principle for bounded-from-below solutions.

Unbounded domains—$m > 2$

Theorem (Barles, R., Quaas, 2020)

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $u_0 \in C(\mathbb{R}^N)$ are bounded from below. Then, there exists a unique, nonnegative, continuous solution of (VHJ).

- *Existence:* aproximation by solutions on bounded domains, compactness arguments.
- *Uniqueness:* comparison principle for bounded-from-below solutions.
- Once existence is proven, we may assume $f, u_0 \geq 0$ with no loss of generality:

$$u \mapsto u + C_1 t + C_2 \quad \text{gives} \quad f \mapsto f + C_1,\ u_0 \mapsto u_0 + C_2$$

.

The ergodic problem in $\mathbb{R}^N$

LTB will again be determined by the *ergodic problem*,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N \tag{E_λ}$$

where both $\lambda \in \mathbb{R}$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ are unknown.

The ergodic problem in $\mathbb{R}^N$

LTB will again be determined by the *ergodic problem*,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N \tag{E_λ}$$

where both $\lambda \in \mathbb{R}$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ are unknown.

- For $\lambda \leq \inf_{\mathbb{R}^N} f$, there exist multiple solutions (constant functions).

The ergodic problem in $\mathbb{R}^N$

LTB will again be determined by the *ergodic problem*,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N \tag{E_λ}$$

where both $\lambda \in \mathbb{R}$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ are unknown.

- For $\lambda \leq \inf_{\mathbb{R}^N} f$, there exist multiple solutions (constant functions).
- *Generalized ergodic constant* is defined as

  $$\lambda^* := \sup\{\lambda \in \mathbb{R} \mid \exists\psi \in C^2(\mathbb{R}^N), \lambda - \Delta\psi + |D\psi|^m \leq f(x)\};$$

  this is analogous to the definition of the generalized principal eigenvalue in [Berestycki et al., 1994].

The ergodic problem in $\mathbb{R}^N$

LTB will again be determined by the *ergodic problem*,

$$\lambda - \Delta\phi + |D\phi|^m = f(x) \quad \text{in } \mathbb{R}^N \tag{E_λ}$$

where both $\lambda \in \mathbb{R}$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ are unknown.

- For $\lambda \leq \inf_{\mathbb{R}^N} f$, there exist multiple solutions (constant functions).
- *Generalized ergodic constant* is defined as

  $$\lambda^* := \sup\{\lambda \in \mathbb{R} \mid \exists\psi \in C^2(\mathbb{R}^N), \lambda - \Delta\psi + |D\psi|^m \leq f(x)\};$$

  this is analogous to the definition of the generalized principal eigenvalue in [Berestycki et al., 1994].
- If f is coercive, there exists a bounded from below solution of (E_{λ^*})

The ergodic problem in $\mathbb{R}^N$ in the case $m > 2$

Theorem ([Barles and Meireles, 2016])

Assume that $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ *is coercive. If* ϕ *is a solution of* (E_{λ_1}) *and* ψ *is a solution of* (E_{λ_2})*, both bounded from below, then* $\lambda_1 = \lambda_2$ *and there exists a constant* $c \in \mathbb{R}$ *such that* $\phi = \psi + c$*.*

- In short, if $m > 2$ there is a unique solution pair (λ^*, ϕ) in the class of bounded from below solutions.
- ϕ is unique up to additive constants, we may assume $\inf_{\mathbb{R}^N} \phi = 0$.
- Uniqueness follows from a comparison principle on exterior domains, i.e., $\mathbb{R}^N \backslash B_R$ for large $R > 0$.

LTB, assumptions on f case $m > 2$

- There exists an increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ and constants $\alpha, \varphi_0, f_0 > 0$ such that for all $r \geq 0$,

$$\varphi_0^{-1} r^{\alpha} \leq \varphi(r)$$

and for all $x \in \mathbb{R}^N$ and $r = |x|$,

$$f_0^{-1} \varphi(r) \leq f(x) \leq f_0(\varphi(r) + 1). \tag{H1}$$

LTB, convergence result case $m > 2$

Theorem (Barles, R., Quaas)

Assuming (H1), *there exists* $\hat{c} \in \mathbb{R}$ *depending on* f *and* u_0 *such that*

$$u(x,t) - \lambda^* t \to \phi(x) + \hat{c} \quad \textit{locally uniformly in } \mathbb{R}^N \textit{ as } t \to +\infty.$$

- No assumptions on the behaviour of u_0 as $|x| \to +\infty$; in particular, it might be very different from that of ϕ.

LTB, convergence result case $m > 2$

Theorem (Barles, R., Quaas)

Assuming (H1)*, there exists* $\hat{c} \in \mathbb{R}$ *depending on* f *and* u_0 *such that*

$$u(x,t) - \lambda^* t \to \phi(x) + \hat{c} \quad \textit{locally uniformly in } \mathbb{R}^N \textit{ as } t \to +\infty.$$

- No assumptions on the behaviour of u_0 as $|x| \to +\infty$; in particular, it might be very different from that of ϕ.
- In [Ichihara, 2012], this is shown for $f(x) \approx |x|^\beta$, where $\beta \geq m^* = \frac{m}{m-1}$, and u_0 has at most polynomial growth.

Elements of the proof case $m > 2$

- For comparison in (VHJ), we use the Hopf-Cole transform $z(x,t) = -e^{-u(x,t)}$ and obtain bounded sub- and super solutions of
$$z_t - \Delta z + N(x, z, Dz) = 0,$$
where $N(x, r, p) = r\left(f(x) + \left|\frac{Dz}{z}\right|^2 - \left|\frac{Dz}{z}\right|^m\right)$.

Elements of the proof case $m > 2$

- For comparison in (VHJ), we use the Hopf-Cole transform $z(x,t) = -e^{-u(x,t)}$ and obtain bounded sub- and super solutions of
$$z_t - \Delta z + N(x, z, Dz) = 0,$$
where $N(x, r, p) = r\left(f(x) + \left|\frac{Dz}{z}\right|^2 - \left|\frac{Dz}{z}\right|^m\right)$.
- We use an ODE approach to construct sub- and super solutions of
$$v_t - \Delta v + |Dv|^m = f(x) - \lambda^*;$$

Behavior of ϕ

Lemma

$$\lim_{|x|\to\infty} \frac{\phi(x)}{|x|} = \infty$$

Proof. Blow up argument to find a supersolution of the eikonal equation Comparison with eikonal equation we find a contradiction.

Lemma (Sub- and supersolutions)

1. *There exists a constant* $\sigma > 0$ *such that* $U(\cdot, t) \to \phi - \sigma$ *and* $V(\cdot, t) \to \phi + \sigma$ *locally uniformly in* $\mathbb{R}^N$ *as* $t \to \infty$.
2. *For any fixed* $\hat{t} > 0$,

$$V(x, \hat{t}) \to +\infty \quad \text{as } x \to \partial Q_{\hat{t}} \ (x \in Q_{\hat{t}}).$$

3. *There exists* $M > 0$ *such that, for all* $t > 0$,

$$U(x, t) \leq t + M \quad \text{for all } x \in \mathbb{R}^N.$$

The Supersolution

$$V(x,t) = \phi(x) + \chi(\phi(x) - (t + t_0)) + \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau.$$

$$\begin{cases} \chi'' = C(\chi')^{\beta_1}(1 + \chi')^{\beta_2} & \text{in } (-\infty, b), \\ \chi(0) = \chi'(0) = 0, & \\ \chi(s) \equiv +\infty & \text{for all } s \geq b. \end{cases} \tag{SUP-ODE}$$

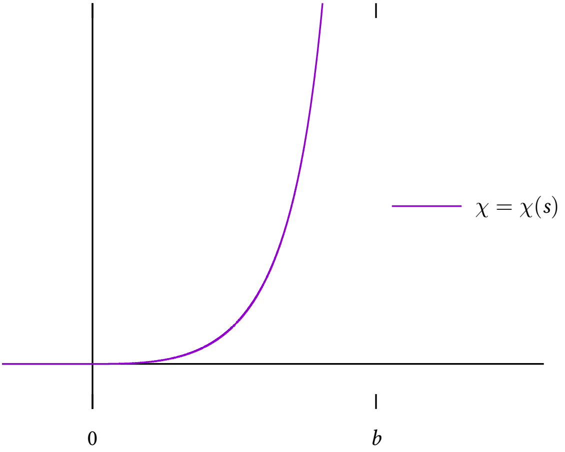
Choice of constants $t_0 > 0$ (to be chosen) determines "when the supersolution comes into play".

Suitable choices of $\beta_1 \in (0,1)$, $\beta_1 + \beta_2 > 1$ imply that (SUP-ODE) has a nontrivial solution and $\hat{\beta} = \hat{\beta}(\beta_1, \beta_2, p)$ is large, hence

$$\sigma := \int_0^\infty (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau < \infty.$$

The Supersolution

$$V(x,t) = \phi(x) + \chi(\phi(x) - (t + t_0)) + \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}}\, d\tau.$$



The Subsolution

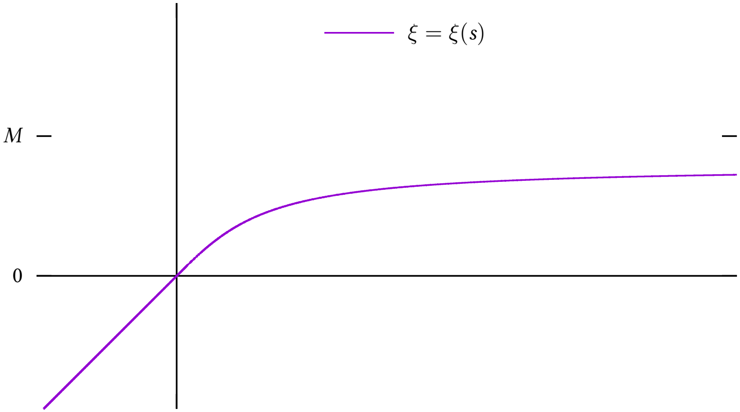
$$U(x,t) = t + t_0 + \xi(\phi(x) - (t + t_0)) - \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau.$$

$$\begin{cases} \xi'' = -C(1-\xi)^{\eta_1}(\xi')^{\eta_2} \quad \text{in } (0,\infty), \\ \xi(0) = 0, \ \xi'(0) = 1, \end{cases} \tag{SUB-ODE}$$

with $t_0, \eta_1, \eta_2 > 0$ to be chosen; $\hat{\alpha}, \hat{\beta} > 0$, as before.

The Subsolution

$$U(x,t) = t + t_0 + \xi(\phi(x) - (t + t_0)) - \int_0^t (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau.$$



Convergence argument

Lemma

$u(x,t) - \lambda^* t$ is bounded over compact sets, uniformly with respect to $t > 0$.

The Lemma allows us to define upper and lower limits at $t \to +\infty$ for $u(x,t) - \lambda^* t$.

Together with local gradient bounds [Barles 2017] by compactness we have convergence along subsequences $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$ over compact sets.

The SMP implies the limit is $\phi + \hat{c}$, for some $\hat{c} \in \mathbb{R}$.

Full convergence

To prove convergence on $\widehat{K} \subset \mathbb{R}^N$ we use

- Convergent subsequence on $\overline{B}_R$ for $B_R \supset \widehat{K}$, with $R >> 1$
- Finer parametrization of sub-, supersolutions U_R, V_R:
  For $R > 0$ and $(x, t) \in \mathbb{R}^N \times (0, +\infty)$,

  $$V_R(x,t) = \phi(x) + \hat{c} + \chi(\phi(x) + \hat{c} - (t+R)) + \int_R^{t+R} (\tau^{\hat{\alpha}} + 1)^{-\hat{\beta}} \, d\tau + \frac{1}{R},$$

  and similarly for a subsolution U_R.
- Note that the “extra terms” vanish as $R \to +\infty$.

Setting for $1 < m \leq 2$

- Existence of solutions for (VHJ), (E_λ), holds roughly as before.

Setting for $1 < m \leq 2$

- Existence of solutions for (VHJ), (E_λ), holds roughly as before.
- No comparison principle, nor uniqueness of solutions for (VHJ) for f as general as before.

Setting for $1 < m \leq 2$

- Existence of solutions for (VHJ), (E_λ), holds roughly as before.
- No comparison principle, nor uniqueness of solutions for (VHJ) for f as general as before.
- Convergence result for *any* solution of (VHJ)

Setting for $1 < m \leq 2$

- Existence of solutions for (VHJ), (E_λ), holds roughly as before.
- No comparison principle, nor uniqueness of solutions for (VHJ) for f as general as before.
- Convergence result for *any* solution of (VHJ)
- Uniqueness of solution pairs for (E_λ) given by [Arapostathis et al., 2019] by dynamical systems arguments.

Setting for $1 < m \leq 2$

- Existence of solutions for (VHJ), (E_λ), holds roughly as before.
- No comparison principle, nor uniqueness of solutions for (VHJ) for f as general as before.
- Convergence result for *any* solution of (VHJ)
- Uniqueness of solution pairs for (E_λ) given by [Arapostathis et al., 2019] by dynamical systems arguments.
- We obtain partial results towards uniqueness for (E_λ) even for sub-solutionby "PDE techniques", but there is no comparison principle for (E_λ) !!

LTB, assumptions on f for the case $1 < m \le 2$

- There exists a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ and $c > 0$ such that, if $r = |x|$, then

$$c^{-1}\varphi(r) \le f(x) \le c(\varphi(r) + 1)$$

  and for sufficiently large $\rho > 0$,

$$\rho\varphi(\rho + 1)^{\frac{1}{m}} \le \varphi(\rho). \tag{H2}$$

- $$\limsup_{x\to+\infty} \frac{|Df(x)|^{\frac{1}{2m-1}}}{|f(x)|^{\frac{1}{m}}} < +\infty \tag{H3}$$

- The initial data satisfies

$$u_0(x) \le c_0 + c_2|x| \left[\inf_{\mathbb{R}^N \setminus B_{\frac{1}{2}|x|}} f\right]^{\frac{1}{m}} \tag{H4}$$

  for a precise value of $c_2 > 0$ and some $c_0 \ge 0$.

Proposition

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ *is coercive. If* χ *and* $v \in USC(\mathbb{R}^N)$ *are respectively a solution and a subsolution of* (E_{λ^*})*, both bounded from below, then there exists* $c \in \mathbb{R}$ *such that* $v(x) = \chi(x) + c$ *for all* $x \in \mathbb{R}^N$*.*

Proposition

Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is coercive. If χ and $v \in USC(\mathbb{R}^N)$ are respectively a solution and a subsolution of (E_{λ^*}), both bounded from below, then there exists $c \in \mathbb{R}$ such that $v(x) = \chi(x) + c$ for all $x \in \mathbb{R}^N$.

The result follows from approximation of (E_λ) over bounded domains and a well chosen perturbation of solutions

LTB, convergence result $1 < m \leq 2$

Theorem (Quaas, R.)

Assume (H2)-(H4) *hold. Then,* $u(\cdot, t) - \lambda^* t \to \phi + \hat{c}$ *locally uniformly over* $\mathbb{R}^N$*, for some constant* $\hat{c} \in \mathbb{R}$*, where* $u = u(x, t)$ *is any solution of* (VHJ) *and* ϕ *is the unique (normalized) solution of* (E_{λ^*})*.*

Elements of the proof $\quad 1 < m \leq 2$

- Supersolutions are given by

$$\begin{cases} \lambda_R - \Delta\phi_R + |D\phi_R|^m = f(x) & \text{in } B_R \\ \lambda_R - \Delta\phi_R + |D\phi_R|^m \geq f(x) & \text{on } \partial B_R, \end{cases}$$

Recall we have $\phi_R(x) \to +\infty$ as $x \to \partial B_R$.

Elements of the proof $1 < m \leq 2$

- Supersolutions are given by

$$\begin{cases} \lambda_R - \Delta\phi_R + |D\phi_R|^m = f(x) & \text{in } B_R \\ \lambda_R - \Delta\phi_R + |D\phi_R|^m \geq f(x) & \text{on } \partial B_R, \end{cases}$$

  Recall we have $\phi_R(x) \to +\infty$ as $x \to \partial B_R$.

- Subsolutions are given by

$$\nu_R + \Delta\psi_R + |D\psi_R|^m = f_R \quad \text{in } \mathbb{R}^N/2S_R\mathbb{Z}^N,$$

  where f_R is the periodic extension of $\min\{f, R\}$ to $\mathbb{R}^N/2S_R\mathbb{Z}^N$ for a suitable S_R.

Elements of the proof $1 < m \leq 2$

- Supersolutions are given by

$$\begin{cases} \lambda_R - \Delta\phi_R + |D\phi_R|^m = f(x) & \text{in } B_R \\ \lambda_R - \Delta\phi_R + |D\phi_R|^m \geq f(x) & \text{on } \partial B_R, \end{cases}$$

  Recall we have $\phi_R(x) \to +\infty$ as $x \to \partial B_R$.

- Subsolutions are given by

$$\nu_R + \Delta\psi_R + |D\psi_R|^m = f_R \quad \text{in } \mathbb{R}^N/2S_R\mathbb{Z}^N,$$

  where f_R is the periodic extension of $\min\{f, R\}$ to $\mathbb{R}^N/2S_R\mathbb{Z}^N$ for a suitable S_R.

- Comparison/maximum principle arguments are made at the level of bounded domains or for (periodic) bounded solutions.

Elements of the proof $1 < m \leq 2$

- Supersolutions are given by

$$\begin{cases} \lambda_R - \Delta\phi_R + |D\phi_R|^m = f(x) & \text{in } B_R \\ \lambda_R - \Delta\phi_R + |D\phi_R|^m \geq f(x) & \text{on } \partial B_R, \end{cases}$$

  Recall we have $\phi_R(x) \to +\infty$ as $x \to \partial B_R$.

- Subsolutions are given by

$$\nu_R + \Delta\psi_R + |D\psi_R|^m = f_R \quad \text{in } \mathbb{R}^N/2S_R\mathbb{Z}^N,$$

  where f_R is the periodic extension of $\min\{f, R\}$ to $\mathbb{R}^N/2S_R\mathbb{Z}^N$ for a suitable S_R.

- Comparison/maximum principle arguments are made at the level of bounded domains or for (periodic) bounded solutions.
- We have $(\lambda_R, \phi_R), (\nu_R, \psi_R) \to (\lambda^*, \phi)$ as $R \to +\infty$.

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).
- LTB ? All space?

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).
- LTB ? All space?
- Nonlocal Periodic setting LTB Barles , Koike Ley, Topp. 2014

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).
- LTB ? All space?
- Nonlocal Periodic setting LTB Barles , Koike Ley, Topp. 2014
- Nonlocal ergodic problem? Dávila, Q. Topp (bounded domain) not clear not uniqueness even in the coercive case.

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).
- LTB ? All space?
- Nonlocal Periodic setting LTB Barles , Koike Ley, Topp. 2014
- Nonlocal ergodic problem? Dávila, Q. Topp (bounded domain) not clear not uniqueness even in the coercive case.
- Nonlocal ergodic all space: Biswas Topp.

Other results and Open problems

- I. Birindelli, F. Demengel, F. Leoni. 2017. Ergodic pairs for singular or degenerate fully nonlinear operators (bounded domain).
- LTB ? All space?
- Nonlocal Periodic setting LTB Barles , Koike Ley, Topp. 2014
- Nonlocal ergodic problem? Dávila, Q. Topp (bounded domain) not clear not uniqueness even in the coercive case.
- Nonlocal ergodic all space: Biswas Topp.
- LTB ?

Thank!

Arapostathis, A., Biswas, A., and Caffarelli, L. (2019).
On uniqueness of solutions to viscous hjb equations with a subquadratic nonlinearity in the gradient.
Communications in Partial Differential Equations, 44(12):1466–1480.

Barles, G. and Da Lio, F. (2004).
On the generalized Dirichlet problem for viscous Hamilton–Jacobi equations.
Journal de Mathématiques Pures et Appliquées, 83(1):53–75.

Barles, G., Ley, O., Nguyen, T.-T., and Phan, T. (2017).
Large time behavior of unbounded solutions of first-order Hamilton–Jacobi equations in $\mathbb{R}^n$.
arXiv preprint arXiv:1709.08387.

Barles, G. and Meireles, J. (2016).
On unbounded solutions of ergodic problems in $\mathbb{R}^m$ for viscous Hamilton–Jacobi equations.
Communications in Partial Differential Equations, 41(12):1985–2003.

Barles, G., Porretta, A., and Tchamba, T. T. (2010).
On the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton–Jacobi equations.
Journal de mathématiques pures et appliquées, 94(5):497–519.

Barles, G., Quaas, A., and Rodríguez-Paredes, A. (2020).
Large-time behavior of unbounded solutions of viscous hamilton-jacobi equations in rn.
Communications in Partial Differential Equations, 46(3):547–572.

Barles, G. and Roquejoffre, J.-M. (2006).
Ergodic type problems and large time behaviour of unbounded solutions of Hamilton–Jacobi equations.
Communications in Partial Differential Equations, 31(8):1209–1225.

Barles, G. and Souganidis, P. E. (2001).
Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations.
SIAM Journal on Mathematical Analysis, 32(6):1311–1323.

Benachour, S., Karch, G., and Laurençot, P. (2004).
Asymptotic profiles of solutions to viscous hamilton–jacobi equations.
Journal de mathématiques pures et appliquées, 83(10):1275–1308.

Berestycki, H., Nirenberg, L., and Varadhan, S. S. (1994).
The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.
Communications on Pure and Applied Mathematics, 47(1):47–92.

Biler, P., Karch, G., and Guedda, M. (2004).
Asymptotic properties of solutions of the viscous hamilton-jacobi equation.
Journal of Evolution Equations, 4(1):75–97.

Friedman, A. (2013).
Partial differential equations of parabolic type.
Courier Corporation.

Gallay, T. and Laurençot, P. (2007).
Asymptotic behavior for a viscous hamilton-jacobi equation with critical exponent.
Indiana University mathematics journal, pages 459–479.

Grenon, N., Murat, F., and Porretta, A. (2013).
A priori estimates and existence for elliptic equations with gradient dependent terms.
Ann. Sc. Norm. Super. Pisa Cl. Sci, 13:137–205.

Iagar, R. G., Laurençot, P., and Vázquez, J. L. (2010).
Asymptotic behaviour of a nonlinear parabolic equation with gradient absorption and critical exponent.
arXiv preprint arXiv:1002.2094.

Ichihara, N. (2012).
Large time asymptotic problems for optimal stochastic control with superlinear cost.
Stochastic Processes and their Applications, 122(4):1248–1275.

Ichihara, N. and Ishii, H. (2009).
Long-time behavior of solutions of hamilton–jacobi equations with convex and coercive hamiltonians.
Archive for rational mechanics and analysis, 194(2):383–419.

Ishii, H. (2008).
Asymptotic solutions for large time of hamilton-jacobi equations in euclidean n space.
In *Annales de l'IHP Analyse non linéaire*, volume 25, pages 231–266.

Ishii, H. (2009).
Asymptotic solutions of hamilton-jacobi equations for large time and related topics.
In *ICIAM 07—6th International Congress on Industrial and Applied Mathematics*, pages 193–217.

Kardar, M., Parisi, G., and Zhang, Y.-C. (1986).
Dynamic scaling of growing interfaces.
Physical Review Letters, 56(9):889.

Lasry, J.-M. and Lions, P.-L. (1989).
Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints.
Mathematische Annalen, 283(4):583–630.

Laurençot, P. (2009).
Non-diffusive large time behavior for a degenerate viscous hamilton–jacobi equation.
Communications in Partial Differential Equations, 34(3):281–304.

Namah, G. and Roquejoffre, J.-M. (1999).
Remarks on the long time behaviour of the solutions of hamilton-jacobi equations.
Communications in partial differential equations, 24(5-6):883–893.

Quittner, P. and Souplet, P. (2007).
Superlinear parabolic problems: blow-up, global existence and steady states.
Springer Science & Business Media.

Souplet, P. and Zhang, Q. S. (2006).
Global solutions of inhomogeneous Hamilton-Jacobi equations.
Journal d'analyse mathématique, 99(1):355–396.

Tabet Tchamba, T. (2010).
Large time behavior of solutions of viscous Hamilton–Jacobi equations with superquadratic Hamiltonian.
Asymptotic Analysis, 66(3-4):161–186.