The Landis conjecture via Liouville comparison principle and criticality theory

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Mostly Maximum Principle

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Landis conjecture (late 60s)

Conjecture (Landis (late 60s))

Let $\Omega = \mathbb{R}^N$ *or an exterior domain, and* $\|V\|_{L^\infty(\Omega)} \leq 1$.
If u solves the Schrödinger equation

$$(-\Delta + V)\varphi = 0 \qquad \text{in } \Omega, \quad \text{and}$$

$u(x) = O(\mathrm{e}^{-k|x|})$ *as* $x \to \infty$*, for some* $k > 1$*, then* $u = 0$.

Historical review

- Meshkov (1991) disproved the conjecture for a complex-valued potential V and $N = 2$ with $u = \mathrm{e}^{-c|x|^{4/3}}$. Moreover, $u = 0$ if $u = o(\mathrm{e}^{-|x|^{4/3+\varepsilon}})$. (cf. Froese-Herbst-2Hoffmann-Ostenhoff (82))

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- What about the case of real-valued V?
- Bourgain/Kenig (2005) $N=2$. $u=o(\mathrm{e}^{-c|x|^{4/3}\log|x|}) \Longrightarrow u=0$.
- Kenig (2005) asked a weaker question: $u=o(\mathrm{e}^{-|x|^{1+\varepsilon}}) \Rightarrow u=0$?
- Kenig/Silvestre/Wang ('15) $N=2$, $V\geq 0$, $u=o(\mathrm{e}^{-c|x|\log|x|}) \Rightarrow u=0$.
- $N=2$, general $\mathcal{L}$: Davey, Davey/Kenig/Wang, Kenig/Wang, Logunov/Malinnikova/Nadirashvili/Nazarov,
- $N\geq 2$, general $\mathcal{L}$. Arapostathis/Biswas/Ganguly, L. Rossi (proved the 1d and radial cases), Sirakov/Souplet.
- For $n\geq 3$ almost all the results assume that $\mathcal{L}\geq 0$ at least outside a compact set.
- Results in the discrete setting: Fernández-Bertolin/Roncal/Stan, Jaming/Lyubarskii/Malinnikova/Perfekt, and Das/Keller/Pinchover.

Preliminaries

Definition (Nonnegativity of operators and (sub)criticality)

Let $\Omega \subset \mathbb{R}^N$ be a domain. An elliptic operator $\mathcal{L}$ of the divergence form

$$\mathcal{L}\varphi := -\mathrm{div}\left[\left(A(x)\nabla\varphi + \varphi\tilde{b}(x)\right)\right] + b(x)\cdot\nabla\varphi + c(x)\varphi, \qquad x \in \Omega,$$

is *nonnegative in* Ω (in short, $\mathcal{L} \geq 0$) if the equation $\mathcal{L}\varphi = 0$ admits a weak positive (super)solution in Ω.

$\mathcal{L} \geq 0$ is *subcritical* in Ω if there exists $W \gneq 0$ such that $\mathcal{L} - W \geq 0$ in Ω. Such a function W is called a *Hardy-weight* for $\mathcal{L}$ in Ω. The operator $\mathcal{L}$ is *critical* in Ω if $\mathcal{L} \geq 0$ in Ω, but $\mathcal{L}$ is not subcritical in Ω.

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$\mathcal{L}$ is subcritical $\iff$ $\mathcal{L}$ admits minimal positive Green function;
$\mathcal{L}$ is critical $\iff$ $\mathcal{L}$ admits an Agmon ground state $\iff$ $\mathcal{L}$ admits a unique positive supersolution (strong positive Liouville theorem).

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Both funtctions are positive solutions of minimal growth near ∞_Ω, where ∞_Ω is the ideal point in the one-point compactification of Ω.

Liouville comparison principle (YP, 2007)

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a domain and

$$H_k := P_k + V_k = -\nabla \cdot (A_k \nabla) + V_k\,, \qquad k = 1, 2,$$

be nonnegative Schrödinger-type operators in Ω, where A_k are symmetric, locally bounded and locally uniformly elliptic matrices.

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be nonnegative Schrödinger-type operators in Ω*, where* A_k *are symmetric, locally bounded and locally uniformly elliptic matrices.*

Assume that the following assumptions hold true.

(i) *The operator* H_1 *is critical in* Ω *with its Agmon ground state* Ψ_{H_1}*.*

(ii) *There exists* $\Phi_2 \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *such that* $H_2\Phi_2 \leq 0$ *in* Ω*, and* $\Phi_2^+ \neq 0$*.*

(iii) $(\Phi_2^+)^2(x) A_2(x) \leq C \Psi_{H_1}^2(x) A_1(x) \quad$ *a.e. in* Ω*.*

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(iii) $(\Phi_2^+)^2(x)A_2(x) \leq C\Psi^2_{H_1}(x)A_1(x)$ *a.e. in* Ω.

Then H_2 *is critical in* Ω, *and* $\Phi_2 > 0$ *is its Agmon ground state.*

Landis-type theorem in the symmetric case

Theorem

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$. Consider two nonnegative Schrödinger-type operators

$$H_k = P_k + V_k = -\mathrm{div}(A_k \nabla) + V_k \geq 0, \quad k = 1, 2.$$

.
- *Suppose that H_1 is critical in Ω with an Agmon ground state Ψ_{H_1}.*
- *Let $W \geq V_2$ outside a compact set in Ω.*
- *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of the equation $H_2 \varphi = 0$ in Ω, s.t.*

$$|u|^2 A_2 \leq C \Psi_{H_1}^2 A_1 \;\; \text{in } \Omega, \quad \text{and} \quad \liminf_{x \to \infty_\Omega} \frac{|u(x)|}{G_{P_2+W}(x)} = 0,$$

where G_{P_2+W} is a positive solution of minimal growth of $(P_2 + W)\varphi = 0$ in a neighborhood of ∞_Ω.

Then $u = 0$.

Proof.

WLOG, assume that $u^+ \neq 0$. By the Liouville comparison principle, H_2 is critical and $u = u^+ > 0$ is its Agmon ground state.
Now, since $u > 0$ and $W \geq V_2$ near ∞_Ω, it follows that

$$(P_2 + W)u = H_2 u + (W - V_2)u = (W - V_2)u \geq 0 \ \text{ near } \infty_\Omega.$$

So, u is a positive supersolution of $(P_2 + W)\varphi = 0$ near ∞_Ω. As G_{P_2+W} is a positive solution of minimal growth near ∞_Ω of the same equation, we conclude that $u \geq CG_{P_2+W}$ near ∞_Ω for some $C > 0$. This contradicts our assumption

$$\liminf_{x \to \infty_\Omega} \frac{|u(x)|}{G_{P_2+W}(x)} = 0.$$

Therefore, $u = 0$. □

Back to Landis Conjecture in $\mathbb{R}^N$

Corollary

Let $H = -\Delta + V$ *be a nonnegative Schrödinger operator in* $\mathbb{R}^N$*, where* $V \leq 1$*. If* $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ *is a solution of the equation* $H\varphi = 0$ *in* $\mathbb{R}^N$ *satisfying*

$$|u(x)| = \begin{cases} O(1) & N = 1, \\ O\left(|x|^{(2-N)/2}\right) & N \geq 2, \end{cases} \quad \text{as } |x| \to \infty,$$

$$\text{and} \qquad \liminf_{|x|\to\infty} \frac{|u(x)||x|^{(N-1)/2}}{\mathrm{e}^{-|x|}} = 0,$$

then $u = 0$.

The linear nonsymmetric case

Theorem

Let $\mathcal{L}$ be an elliptic operator in divergence form. Assume that $\mathcal{L} \geq 0$ in Ω. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of the equation $\mathcal{L}\varphi = 0$ in Ω such that

$$|u| = O(G_{\mathcal{L}}) \text{ as } x \to \infty_\Omega \quad \text{and} \quad \liminf_{x \to \infty_\Omega} \frac{|u(x)|}{G_{\mathcal{L}}(x)} = 0,$$

where $G_{\mathcal{L}}$ is a positive solution of the equation $\mathcal{L}\varphi = 0$ of minimal growth near ∞_Ω.
Then $u = 0$.

The linear nonsymmetric case (stronger version)

Theorem

Assume that $\mathcal{L} \geq 0$ in Ω, where $\mathcal{L}$ is of divergence form. Let $W \geq 0$ be a critical Hardy-weight for $\mathcal{L}$ with Agmon ground state $\Psi_{\mathcal{L}-W}$, and consider a potential V s.t. $V + W \geq 0$ in Ω.
Let u be a solution of the equation $(\mathcal{L} + V)\varphi = 0$ in Ω such that

$$|u| = O(\Psi_{\mathcal{L}-W}) \ \text{ as } \ x \to \infty_\Omega, \quad \text{and} \quad \liminf_{x \to \infty_\Omega} \frac{|u(x)|}{\Psi_{\mathcal{L}-W}(x)} = 0.$$

Then $u = 0$.

Application

Let $\mathcal{L} = -\Delta + 1 + \mathcal{V}_{\mathrm{sr}}$ *be a subcritical operator in* $\mathbb{R}^N$*, where* $\mathcal{V}_{\mathrm{sr}}$ *is a short range potential. The minimal Green function* $\mathcal{G}_{\mathcal{L}}$ *of* $\mathcal{L}$ *satisfies*

$$\mathcal{G}_{\mathcal{L}}(x) \asymp \frac{\mathrm{e}^{-|x|}}{|x|^{(N-1)/2}} \quad \text{as } |x| \to \infty.$$

Moreover, there exists a positive solution Φ *of* $\mathcal{L}[\varphi] = 0$ *in* $\mathbb{R}^N$ *such that*

$$\Phi(x) \asymp \frac{\mathrm{e}^{|x|}}{|x|^{(N-1)/2}} \quad \text{as } |x| \to \infty.$$

Let $W := \frac{\mathcal{L}(\sqrt{\mathcal{G}_{\mathcal{L}}\Phi})}{\sqrt{\mathcal{G}_{\mathcal{L}}\Phi}}$, $\Psi_{\mathcal{L}-W} := \sqrt{\mathcal{G}_{\mathcal{L}}\Phi} \asymp |x|^{\frac{(1-N)}{2}}$. *Since* $\lim_{|x|\to\infty} \mathcal{G}_{\mathcal{L}}/\Phi = 0$, *hence* $\Psi_{\mathcal{L}-W}$ *is Agmon g.s. of the critical operator* $\mathcal{L} - W$.

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If u *is a solution of* $(\mathcal{L} + V)[\varphi] = 0$ *in* $\mathbb{R}^N$, *where* $V + W \geq 0$ *and*

$$|u(x)| = O\left(|x|^{\frac{(1-N)}{2}}\right) \quad \text{as } |x| \to \infty, \quad \text{and} \quad \liminf_{|x|\to\infty} |u(x)||x|^{\frac{(N-1)}{2}} = 0,$$

then $u = 0$.

Exterior domains

Theorem

Suppose that $\mathcal{L} \geq 0$ in $\Omega \setminus K$ for some $K \Subset \Omega$, and satisfies the unique continuation property in Ω.
Let u be a solution of $\mathcal{L}\varphi = 0$ in Ω such that u has constant sign in a neighborhood of ∂K, and

$$|u| = O(G_{\mathcal{L}}) \ \text{ as } \ x \to \bar{\infty}_{\Omega}, \quad \text{and} \quad \liminf_{x \to \bar{\infty}_{\Omega}} \frac{|u(x)|}{G^{\Omega}_{\mathcal{L}}(x)} = 0\,,$$

where $G^{\Omega}_{\mathcal{L}}$ is a positive solution of the equation $\mathcal{L}\varphi = 0$ of minimal growth at ∞_{Ω}. Then $u = 0$.

Conclusion remarks

- In the selfadjoint case we are left with two challenging cases: either

  (a) $\inf \sigma_{\mathrm{ess}}(H) < 0$ (0 is an embedded eigenvalues), or

  (b) $\inf \sigma_{\mathrm{ess}}(H) = 0$ and the Morse index of H is not finite (H admits infinitely many negative eigenvalues).
- Using a Liouville comparison principle for quasilinear operators of the form

$$\mathcal{Q}[\varphi] := -\mathrm{div}(|\nabla\varphi|_{A(x)}^{p-2} A(x)\nabla\varphi) + V|\varphi|^{p-2}\varphi$$

  we prove Landis-type results for solutions u of $\mathcal{Q}[\varphi] = 0$.
- Together with Matthias Keller and Ujjal Das, we study Landis conjecture for nonnegative Schrödinger-type operators and quasilinear operators on infinite discrete graphs.

Thank you for your attention!