Mostly Maximum Principle

5 th edition: in Latin America for the first time

An upper bound for the least energy of a sign-changing solution to a zero mass problem

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Open problem

Tobias Weth (CVPDE 2006): Energy bounds for entire nodal solutions of autonomous superlinear equations,
considers the autonomous problems:

$$-\Delta u = |u|^{\frac{4}{N-2}} u, \qquad u \in D^{1,2}(\mathbb{R}^N), \quad N \geq 3,$$

and

$$-\Delta u + au = f(u), \qquad u \in H^1(\mathbb{R}^N);$$

Theorem

There is $\varepsilon_\Phi > 0$ such that $J(u) > \frac{2}{N} S^{N/2} + \varepsilon_\Phi$ for every sign changing solution $u \in D^{1,2}(\mathbb{R}^N)$ of first problem above.
Moreover, there is $\varepsilon_\Psi > 0$ such that $J(u) > 2c_0 + \varepsilon_\Psi$ for every sign changing solution $u \in H^1(\mathbb{R}^N)$ of second problem above.

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Moreover, there is $\varepsilon_\Psi > 0$ such that $J(u) > 2c_0 + \varepsilon_\Psi$ for every sign changing solution $u \in H^1(\mathbb{R}^N)$ of second problem above.

We obtain an upper bound:

$$2c_0 + \varepsilon_\Psi < J(u) < 10c_0$$

Least energy of a sign-changing solution to the problem

$$-\Delta u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \tag{1}$$

where $N \geq 1$ and the nonlinearity f is subcritical at infinity and supercritical near the origin.

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- Berestycki and Lions (1983): problem (1) has a ground state solution which is positive, radially symmetric and decreasing in the radial direction.
- Benci and Micheletti (2006), Alves and Souto (2012), Clapp-M. (2018), Clapp, M. and Pellacci (2021) and others: one or multiple positive solutions for a similar equation involving a scalar potential that decays to zero at infinity, both in the whole space and in an exterior domain.
- Mederski (2021) a nonradial sign-changing solution (but its energy is not known).

$$-\Delta u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \quad N \geq 5,$$

(f_1) $f \in \mathcal{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R})$ with $\alpha \in \left(\frac{N}{2(N-2)}, 1\right]$, and there exist $a_1 > 0$ and $2 < p < 2^* := \frac{2N}{N-2} < q$ such that, for $\kappa = -1, 0, 1$,

$$|f^{(\kappa)}(s)| \leq \begin{cases} a_1 |s|^{p-(\kappa+1)} & \text{if } |s| \geq 1, \\ a_1 |s|^{q-(\kappa+1)} & \text{if } |s| \leq 1, \end{cases}$$

where $f^{(-1)} := F$, $f^{(0)} := f$, $f^{(1)} := f'$, and $F(s) := \int_0^s f(t)\mathrm{d}t$.

(f_2) There is a constant $\theta > 2$ such that $0 \leq \theta F(s) \leq f(s)s < f'(s)s^2$ for all $s > 0$.

(f_3) f is odd.

Example: $f(s) = \dfrac{u^{q-1}}{1 + u^{(q-p)}}$, for $2 < p < 2^* < q$.

A nonradial sign-changing solution

Theorem [Clapp-M.-Pellacci 2024]

Assume that f satisfies $(f_1) - (f_3)$. Then, there exists a nonradial sign-changing solution $\widehat{\omega}$ to the problem (1) whose energy satisfies

$$2c_0 < \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \widehat{\omega}|^2 - \int_{\mathbb{R}^N} F(\widehat{\omega}) < \begin{cases} 12c_0 & \text{if } N = 5, 6, \\ 10c_0 & \text{if } N \geq 7, \end{cases}$$

where c_0 is the ground state energy of (1). Furthermore, for each $(z_1, z_2, y) \in \mathbb{R}^N \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$,

(a) $\widehat{\omega}(z_1, z_2, y) = \widehat{\omega}(\mathrm{e}^{2\pi \mathrm{i} j/m} z_1, \mathrm{e}^{2\pi \mathrm{i} j/m} z_2, y)$ for all $j = 0, \ldots, m-1$,

(b) $\widehat{\omega}(z_1, z_2, y) = -\widehat{\omega}(z_2, z_1, y)$,

(c) $\widehat{\omega}(z_1, z_2, y_1) = \widehat{\omega}(z_1, z_2, y_2)$ if $|y_1| = |y_2|$,

with $m = 6$ if $N = 5, 6$ and $m = 5$ if $N \geq 7$, and $\widehat{\omega}$ has least energy among all nontrivial solutions satisfying $(a), (b), (c)$.

Previous upper bounds of energy of sign-changing solution

- Clapp and Srikanth (2016) in the positive mass case, $-\Delta u + u = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N)$, for the subcritical pure power nonlinearity $2 < p < 2^*$, an estimate for the least possible energy of a sign-changing solution:

$$J_\infty(\widehat{w}) < 12c_0.$$

- Clapp and Soares (2023): recently improved

$$J_\infty(\widehat{w}) < 10c_0.$$

- Clapp, Pistoia and Weth (2022): it was shown that the same estimates as in our main Theorem hold true for the critical pure power nonlinearity $f(u) = |u|^{2^*-2}u$.

Existence of a sign-changing solution

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- The symmetries introduced in the work of Mederski, however, have only infinite and trivial orbits. This does not allow us to estimate the energy of the solution.
- Here, in contrast, we consider symmetries given by a finite group. This makes it harder to show existence due to the lack of compactness but, once the existence of a solution is established, one immediately gets an upper estimate for its energy.

The symmetric variational setting

- G is a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and denote by

$$Gx := \{gx : g \in G\} \qquad \text{and} \qquad G_x := \{g \in G : gx = x\}$$

the G-orbit and the G-isotropy group of a point $x \in \mathbb{R}^N$. The G-orbit Gx is G-homeomorphic to the homogeneous space G/G_x. So both have the same cardinality, i.e., $|Gx| = |G/G_x|$.

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- $\phi : G \to \mathbb{Z}_2 := \{-1, 1\}$ is a continuous homomorphism of groups. A function $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$u(gx) = \phi(g)u(x) \quad \text{for all} \quad g \in G, \; x \in \mathbb{R}^N,$$

will be called ϕ-*equivariant*.

The symmetric variational setting

- If $\phi \equiv 1$ is the trivial homomorphism then u is G-invariant $u(gx) = u(x)$, i.e., it is constant on every G-orbit, while if ϕ is surjective and $u \neq 0$ then u is nonradial and changes sign $u(gx) = \phi(g)u(x) = -u(x)$. There are surjective homomorphisms for which the only ϕ-equivariant function is the trivial one, as occurs, for example, when $G = O(N)$ and $\phi(g)$ is the determinant of g.

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(A_ϕ) If ϕ is surjective, then there exists $\zeta \in \mathbb{R}^N$ such that $(\ker\phi)\zeta \neq G\zeta$, where $\ker\phi := \{g \in G : \phi(g) = 1\}$.

If K is a closed subgroup of G we write $\phi|K : K \to \mathbb{Z}_2$ for the restriction of ϕ to K. Note that $\phi|K$ satisfies $(A_{\phi|K})$ if ϕ satisfies (A_ϕ), more precisely, if $\phi|K : K \to \mathbb{Z}_2$ is surjective, then $(\ker(\phi|K))\zeta \neq K\zeta$.

The symmetric variational setting

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\},$$

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \qquad \|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2},$$

and set

$$D^{1,2}(\mathbb{R}^N)^\phi := \{u \in D^{1,2}(\mathbb{R}^N) : u \text{ is } \phi\text{-equivariant}\}.$$

Proposition

If ϕ satisfies (A_ϕ) then $D^{1,2}(\mathbb{R}^N)^\phi$ has infinite dimension.

ϕ-equivariant solutions to the problem

Critical points of the functional $J : D^{1,2}(\mathbb{R}^N)^\phi \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u),$$

where $F(u) := \int_0^u f(s)\,\mathrm{d}s$.

$$J'(u)v = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} f(u)v, \qquad u, v \in D^{1,2}(\mathbb{R}^N)^\phi;$$

The nontrivial ϕ-equivariant solutions belong to the set

$$\mathcal{N}^\phi := \{u \in D^{1,2}(\mathbb{R}^N)^\phi : u \neq 0,\ J'(u)u = 0\},$$

which is a closed $\mathcal{C}^1$-submanifold of $D^{1,2}(\mathbb{R}^N)^\phi$ and a natural constraint for J,

$$c^\phi := \inf_{u \in \mathcal{N}^\phi} J(u) > 0$$

Range of energy for existence of solution

Proposition

If

$$c^{\phi} < |G/G_{\xi}|\, c^{\phi|G_{\xi}} \qquad \text{for every } \ \xi \in \mathbb{R}^N \text{ with } G_{\xi} \neq G,$$

then c^{ϕ} is attained.

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then c^{ϕ} is attained.

This previous result depends crucially on the following:

Lemma [Clapp-M. 2018]

If (u_k) is bounded in $D^{1,2}(\mathbb{R}^N)$ and there exists $R > 0$ such that

$$\lim_{k\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_k|^2\right) = 0,$$

then $\lim_{k\to\infty}\int_{\mathbb{R}^N} f(u_k)u_k = 0.$

An upper bound for the energy of symmetric minimizers

A ground state solution $\omega \in \mathcal{C}^2(\mathbb{R}^N)$ to (1):

$$0 < b_1(1+|x|)^{-(N-2)} \leq \omega(x) \leq b_2(1+|x|)^{-(N-2)},$$
$$|\nabla \omega(x)| \leq b_3(1+|x|)^{-(N-1)}.$$

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$$|\nabla\omega(x)| \le b_3(1+|x|)^{-(N-1)}.$$

Definition

We write $\mathbb{R}^N \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$ and a point in $\mathbb{R}^N$ as $(z_1, z_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$. For $m \in \mathbb{N}$, let

$$\mathbb{Z}_m := \{\mathrm{e}^{2\pi \mathrm{i} j/m} : j = 0, \dots, m-1\},$$

G_m be the group generated by $\mathbb{Z}_m \cup \{\tau\}$, acting on $\mathbb{R}^N$ as

$$\mathrm{e}^{2\pi \mathrm{i} j/m}(z_1, z_2, y) := (\mathrm{e}^{2\pi \mathrm{i} j/m} z_1, \mathrm{e}^{2\pi \mathrm{i} j/m} z_2, y), \quad \tau(z_1, z_2, y) := (z_2, z_1, y),$$

and $\phi : G_m \to \mathbb{Z}_2$ be the homomorphism satisfying $\phi(\mathrm{e}^{2\pi \mathrm{i} j/m}) = 1$ and $\phi(\tau) = -1$. Set $\zeta := (1, 0, 0)$, and for each $R > 1$ define sum on $g \in G_m$

$$\widehat{\sigma}_R(x) := \sum \phi(g)\,\omega(x - Rg\zeta), \qquad x \in \mathbb{R}^N.$$

Properties of $\widehat{\sigma}_R(\cdot)$

- $\widehat{\sigma}_R(gx) = \phi(g)\widehat{\sigma}_R(x)$ for every $g \in G_m$, $x \in \mathbb{R}^N$.
- There exists $R_0 > 0$ and for each $R \geq R_0$ a unique $t_R > 0$ such that

$$\sigma_R := t_R \widehat{\sigma}_R \in \mathcal{N}^{\phi}.$$

- $t_R \to 1$ as $R \to \infty$.

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Proposition

If

$$m \geq \sqrt{2}\pi \left(\frac{\pi}{\sqrt{2}} \right)^{\frac{1}{N-3}} \tag{2}$$

then, for R large enough,

$$c^\phi \leq J(\sigma_R) < 2mc_0,$$

where c_0 is the ground state energy of (1).

Lemma[Clapp-M.-Pellacci 2024]

Let $y_1, \dots, y_n$ be n different points in $\mathbb{R}^N$ and $\vartheta_1, \dots, \vartheta_n \in (0, N)$ be such that $\vartheta := \vartheta_1 + \cdots + \vartheta_n > N$. Then there exists $C = C(\vartheta_i, N) > 0$ such that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{n} (1 + |x - Ry_i|)^{-\vartheta_i} \, \mathrm{d}x \le C d^{-\mu} R^{-\mu}$$

for all $R \ge 1$, where $d := \min\{|y_i - y_j| : i, j = 1, \dots, n, \; i \ne j\}$ and $\mu := \vartheta - N$.

Proof of $c^\phi \le J(\sigma_R) < 2mc_0$

We write the G_m-orbit of $\zeta = (1,0,0)$ as

$$G_m\zeta = \{\zeta_1, \dots, \zeta_{2m}\} \quad \text{with} \quad \zeta_i := \mathrm{e}^{2\pi \mathrm{i}(i-1)/m}\zeta \quad \text{and} \quad \zeta_{m+i} := \tau\zeta_i, \quad i = 1, \dots$$

and set

$$\omega_{iR}(x) := \begin{cases} \omega(x - R\zeta_i) & \text{for} \quad i = 1, \dots, m, \\ -\omega(x - R\zeta_i) & \text{for} \quad i = m+1, \dots, 2m. \end{cases}$$

Then, $\sigma_R = \sum_{i=1}^{2m} t_R \omega_{iR}$; and $J(\omega_{iR}) = J(\omega)$. So

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Then, $\sigma_R = \sum_{i=1}^{2m} t_R\omega_{iR}$; and $J(\omega_{iR}) = J(\omega)$. So

$$\begin{aligned} J(\sigma_R) =& \frac{1}{2}\Big\| \sum_{i=1}^{2m} t_R\omega_{iR} \Big\|^2 - \int_{\mathbb{R}^N} F\Big(\sum_{i=1}^{2m} t_R\omega_{iR} \Big) \\ \le& 2mc_0 - \frac{t_R^2}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{2m} \int_{\mathbb{R}^N} f(\omega_{iR})\omega_{jR} + C|t_R - 1|R^{2-N} + CR^{-\mu_1} + CR^{-\mu_2}, \end{aligned}$$

Since, by assumption (f_1), $\alpha > \frac{N}{2(N-2)}$, we have that $\mu_1, \mu_2 > N - 2$.

Non-homogeneous nonlinearity

Lemma 1

Given $n \in \mathbb{N}$, $\overline{u} > 0$ and $f \in \mathcal{C}^{1,\beta}_{\mathrm{loc}}(\mathbb{R})$ with $\beta \in (0,1]$ such that $f(0) = 0$, there exists $b_1 > 0$ such that for any $u_1, \ldots, u_n \in [-\overline{u}, \overline{u}]$,

$$\left| f\Big(\sum_{i=1}^{n} u_i\Big) - \Big(\sum_{i=1}^{n} f(u_i)\Big) \right| \leq b_1 \sum_{\substack{i,j=1 \\ i<j}}^{n} |u_i u_j|^{\beta}.$$

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Lemma 2

Given $n \in \mathbb{N}$, $\overline{u} > 0$ and $f \in \mathcal{C}^{1,\beta}_{\rm loc}(\mathbb{R})$ with $\beta \in (0,1]$ such that $f(0) = 0 = f'(0)$, there exists $b_2 > 0$ such that

$$\left| F\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} F(u_i) - \sum_{\substack{i,j=1 \\ i\neq j}}^{n} f(u_i) u_j \right| \leq b_2 \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} |u_i u_j|^{1+\frac{\beta}{2}} + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} |u_i u_j|^{\beta} |u_k| \right)$$

for any $u_1, \ldots, u_n \in [-\overline{u}, \overline{u}]$, where $F(u) := \int_0^u f$.

Decay and interaction

Lemma

There are positive constants C_0 and $\widehat{C}_0$ such that

$$\lim_{|y|\to\infty} |y|^{N-2} \int_{\mathbb{R}^N} f(\omega(x))\omega(x-y)\,\mathrm{d}x = C_0$$

$$\lim_{|y|\to\infty} |y|^{N-2} \int_{\mathbb{R}^N} |\omega(x)|^{2^*-1}\omega(x-y)\,\mathrm{d}x = \widehat{C}_0.$$

Next, we estimate the second summand $-\frac{t_R^2}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{2m}\int_{\mathbb{R}^N} f(\omega_{iR})\omega_{jR}$. Set $d_{ij} := |\zeta_i - \zeta_j|$ for $i \neq j$. Note that $d_{12} = 2\sin\left(\frac{\pi}{m}\right)$, and $d_{ij} = \sqrt{2}$ if $1 \le i \le m < j \le 2m$. Therefore,

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{m} d_{ij}^{2-N} - \sum_{i=1}^{m}\sum_{j=m+1}^{2m} d_{ij}^{2-N} \\
&\geq (d_{1\,2}^{2-N} + d_{m\,1}^{2-N}) + \sum_{i=2}^{m-1}(d_{i\,(i+1)}^{2-N} + d_{(i-1)\,i}^{2-N}) \\
&+ (d_{m\,1}^{2-N} + d_{(m-1)\,m}^{2-N}) - \sum_{i=1}^{m}\sum_{j=m+1}^{2m} d_{ij}^{2-N} \\
&= 2m\Big(2\sin\left(\frac{\pi}{m}\right)\Big)^{2-N} - m^2(\sqrt{2}\,)^{2-N} \\
&> m\Big[2\Big(\frac{2\pi}{m}\Big)^{2-N} - m(\sqrt{2}\,)^{2-N}\Big] \geq 0
\end{aligned}
$$

because, by assumption,

$$
m \geq \sqrt{2}\pi\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{1}{N-3}}.
$$

For $C_0 > 0$,

$$M_0 := 2(C_0 - \varepsilon) \sum_{\substack{i,j=1 \\ i \neq j}}^{m} d_{ij}^{2-N} - 2(C_0 + \varepsilon) \sum_{i=1}^{m} \sum_{j=m+1}^{2m} d_{ij}^{2-N} > 0.$$

Then, for R large enough we have that

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{2m} \int_{\mathbb{R}^N} f(\omega_{iR})\omega_{jR} = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \int_{\mathbb{R}^N} f(\omega)\omega(\cdot - R(\zeta_j - \zeta_i)) - 2 \sum_{i=1}^{m} \sum_{j=m+1}^{2m} \int_{\mathbb{R}^N} f(\omega)\omega(\cdot - R(\zeta_j - \zeta_i)) \geq M_0 R^{2-N},$$

and we derive that

$$J(\sigma_R) \leq 2mc_0 - \frac{t_R^2}{2} M_0 R^{2-N} + C|t_R - 1| R^{2-N} + o(R^{2-N}).$$

Since $M_0 > 0$ and $t_R \to 1$ as $R \to \infty$, we conclude that $J(\sigma_R) < 2mc_0$ for R large enough, as claimed.

Conclusion

Summary

The function $\psi(t) := \sqrt{2}\pi\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{1}{t}}$ is decreasing in $t > 0$. Since $\psi(t) \to \sqrt{2}\pi$ as $t \to \infty$ and $\sqrt{2}\pi > 4$, any number m satisfying $m \geq \sqrt{2}\pi\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{1}{N-3}}$ must be greater than or equal to 5. Direct computation shows that the least integer greater than or equal to $\sqrt{2}\pi\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{1}{N-3}}$ is 6 if $N = 5, 6$, and it is 5 if $N \geq 7$.


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