A Schiffer-type problem in annuli and applications to Euler flows

Pieralberto Sicbaldi

Joint work with A. Enciso, A. J. Fernández and D. Ruiz

Rio de Janeiro

June 2024

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To find domains $\Omega \subset \mathbb{R}^2$ that support a solution of the overdetermined problem

$$\begin{cases} \Delta u + \mu u = 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \\ u = \text{constant} & \text{on} \quad \partial\Omega \end{cases}$$

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The conjecture is still open.

The Pompeiu problem (1929)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ an (unknown) continuous function and $\Omega \subset \mathbb{R}^2$ a given bounded domain. Is it possible to determine f if we know the values

$$\int_{\mathcal{R}(\Omega)} f(x)\mathrm{d}x$$

for any rigid motion $\mathcal{R}$ in $\mathbb{R}^2$?

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Ellipses and **Polygons** have the Pompeiu property.

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Open problem 1. Is it true that disks are the only domains in $\mathcal{S}$ not having the Pompeiu property?

Open problem 2. Is it true that disks are the only domains in $\mathcal{S}$ where the Schiffer Problem can be solved? (most important case of the Schiffer conjecture)

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5. The fourth or fifth order interior normal derivative of u is constant on $\partial\Omega$ (Kawohl-Lucia, 2020).

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Let $\Omega \subset \mathbb{R}^2$ a smooth domain, and let us denote by Γ_i the connected components of $\partial\Omega$. Assume that there exists a solution to the "Schiffer-type" problem:

$$\begin{cases} \Delta u + \mu u = 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \\ u = c_i & \text{on} \quad \partial\Omega \end{cases}$$

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"Question". Is it true that Ω must be a disk or an annulus?

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In this talk we skip the proof of such two results.

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Theorem 2. For many integer numbers l there exist domains $\Omega_{l,s} \subset \mathbb{R}^2$ given in polar coordinates by

$$\Omega_{l,s} := \{(r,\theta) : a_l + s\, b_{l,s}(\theta) < r < 1 + s\, B_{l,s}(\theta)\}\,, \quad s \in (-\epsilon, \epsilon)\,,$$

where $a_l \in (0,1)$ and $b_{l,s}$, $B_{l,s}$ are analytic functions of the form

$$b_{l,s}(\theta) = \alpha_l \cos l\theta + \mathcal{O}(s)\,, \qquad B_{l,s}(\theta) = \beta_l \cos l\theta + \mathcal{O}(s)\,,$$

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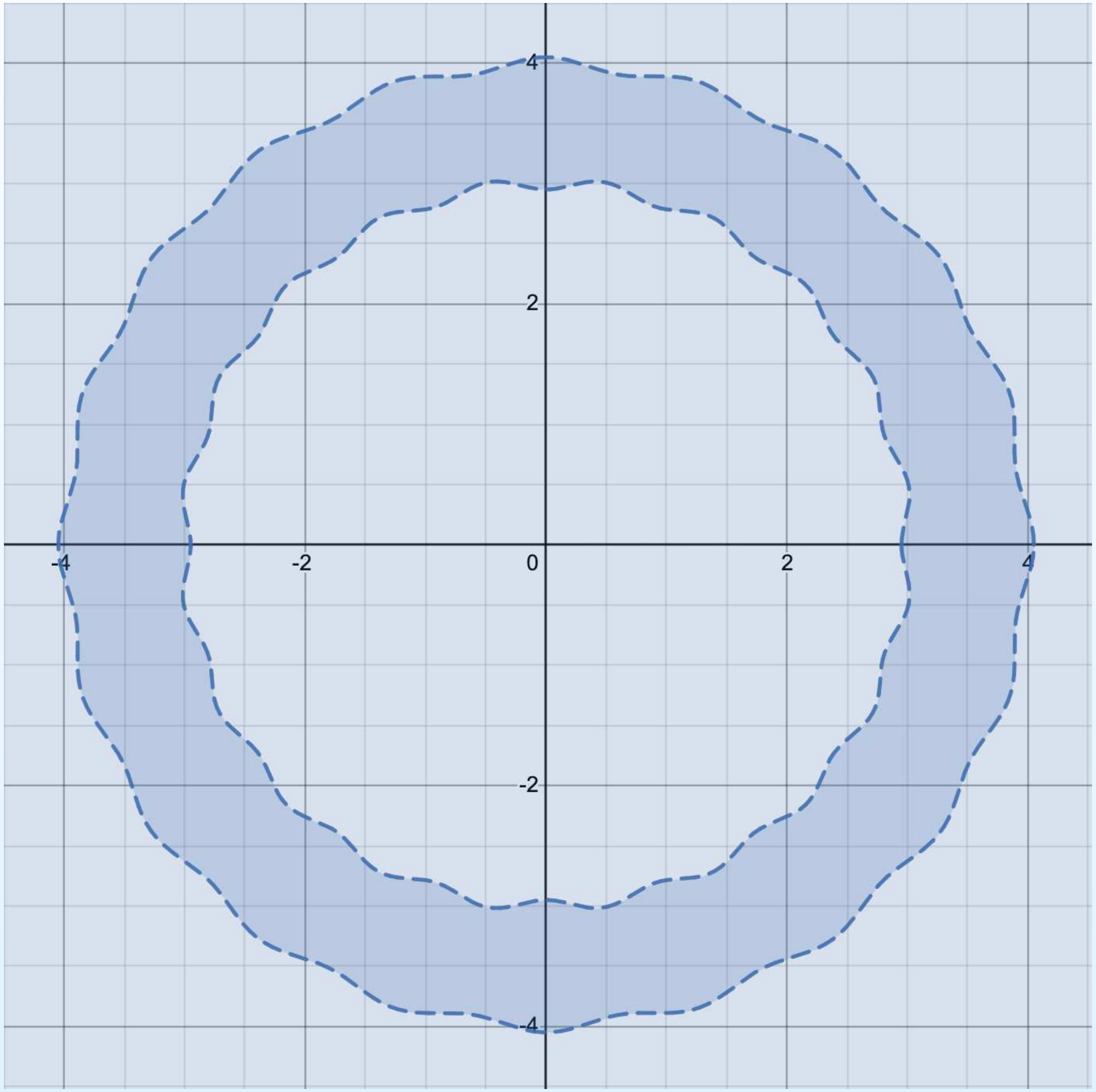
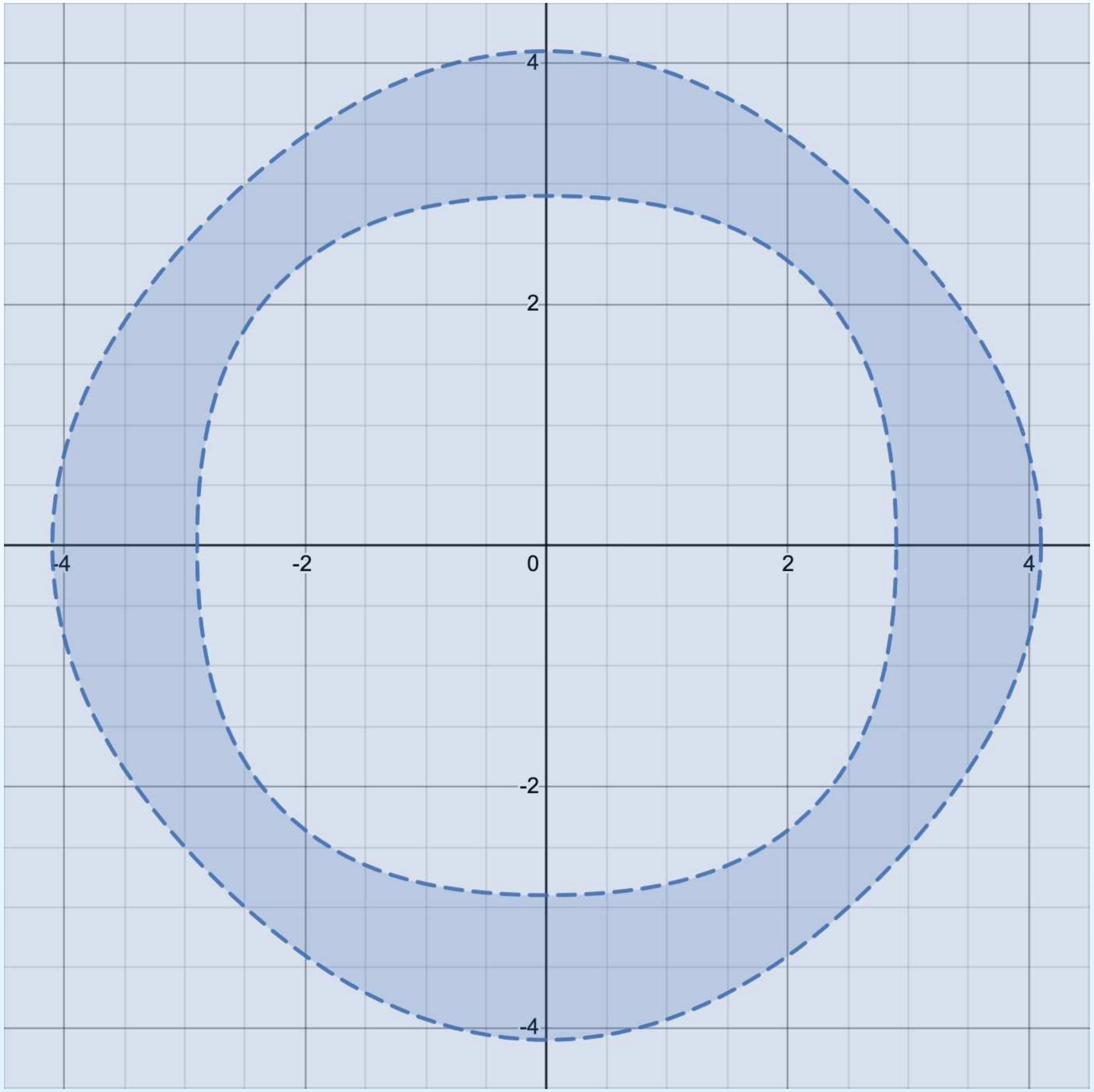
The result holds for example for $l = 4$ and for any l large enough.

The shape of the domains $\Omega_{l,s}$



The case $l = 4$



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The stationary incompressible Euler equations in the plane are:

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \mathbb{R}^2\,, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^2\,. \end{cases}$$

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Take u as given by our main theorem. Then, define

$$v = \begin{cases} (\nabla u)^{\perp} & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 \backslash \Omega, \end{cases} \qquad p = \begin{cases} \mu \left(\frac{c_0^2}{2} - \frac{c_1^2}{2} \right) & \text{in } \Omega_{int}, \\ -\frac{1}{2}(|\nabla u|^2 + \mu(u^2 - c_0^2)) & \text{in } \Omega_{l,s}, \\ 0 & \text{in } \Omega_{ext}\,. \end{cases}$$

where Ω_{int} and Ω_{ext} are the two connected components of $\mathbb{R}^2 \backslash \Omega_{l,s}$ respectively bounded and unbounded.

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Crandall–Rabinowitz Theorem. X, Y = Banach spaces, U neighborhood of 0 in X, Λ real interval. $F : \Lambda \times U \to Y$ C^1 s.t.

i) $F(a, 0) = 0$ for all $a \in \Lambda$;

ii) Ker $D_w F(a_*, 0) = \mathbb{R}\, w_0$ for some $a_* \in \Lambda$ and $w_0 \in X \setminus \{0\}$;

iii) codim Im $D_w F(a_*, 0) = 1$;

iv) $D_a D_w F(a_*, 0)(w_0) \notin$ Im $D_w F(a_*, 0)$ ("transversality").

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If $X = \mathbb{R}\, w_0 \oplus \dot{X}$, there exists a nontrivial smooth curve

$$(-\epsilon, \epsilon) \to \dot{\Lambda} \times X, \quad s \mapsto (a(s), w(s))$$

such that

1) $w(0) = 0$, $a(0) = a_*$, and $s\,(w_0 + w(s)) \in U$;

2) $F(s\,(w_0 + w(s))\,, a(s)) = 0$.

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and the pullback PDE:

$$L_a^{b,B} := (\Phi_a^{b,B})^* \Delta + \mu_2(a)\, Id \ , \ L_a^{b,B}(u) = 0 \text{ in } \Omega \,.$$

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Let $C_l^{k,\alpha}(\Omega)$ the space of $C^{k,\alpha}(\Omega)$ functions with mode l, i.e. of type

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We need also regularity, i.e. very special Hölder spaces:

$X^k := \{u \in C_l^{k,\alpha}(\Omega)\,|\,\partial_r u \in C^{k,\alpha}(\Omega)\}$.

$X_D^k := \{u \in X^k\,|\,u = 0 \text{ on } \partial\Omega\}$.

$X_{DN}^k := \{u \in X^k\,|\,u = 0, \partial_r u = 0 \text{ on } \partial\Omega\}$.

$Y := C_l^{1,\alpha}(\Omega) + X_D^0$, $\quad \|y\|_Y = \inf\{\|y_1\| + \|y_2\|\,|, \, y = y_1 + y_2\}$.

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For $v \in X_D^2$ small we define two functions on $\partial\Omega$

$$b_v(\theta) = c_1(a)\partial_r v\left(\frac{1}{2}, \theta\right) \quad , \quad B_v(\theta) = c_2(a)\partial_r v(1, \theta)\,,$$

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$$w_v(r,\theta) = v(r,\theta) + \frac{\overline{\psi}_a'(r)}{2(1-a)}\Big[2(1-r)b_v(\theta) + (2r-1)B_v(\theta)\Big] \in \mathcal{X}_{DN}^{2,\alpha}\,.$$

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With this, we define the operator:

$$F(a,v) := L_a^{b_v,B_v}[\bar{\psi}_a + w_v] \in Y$$

where $\bar{\psi}_a$ is the pullback of ψ_a for the operator $L_a^{0,0}$.

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Proposition. We have:

- If a is close to 1, then $\mu_2(a) > \lambda_l(a)$
- If a is close to 0 and $l \geq 4$, then $\mu_2(a) < \lambda_l(a)$

So there exist $a = a_*$ such that the two eigenvalues are equal, and there $D_wF(a,0)$ is degenerate.

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$\Longrightarrow$ **all the hypothesis of the Crandall-Rabinowitz theorem**

THANK YOU FOR YOUR ATTENTION