The eigenvalue problem for the fractional generalized Laplacian

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Mostly Maximum Principles

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Merriam-Webster

Mostly: for the greatest part, MAINLY

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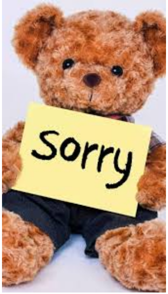
Meaning NOT ALL

$\underbrace{\text{~~Mostly~~}}_{\text{Some}}$ Maximum Principles

~~Mostly~~ Maximum Principles
Some



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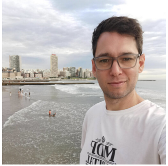
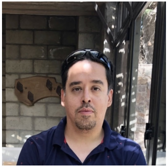


Ariel Salort



Juan Spedaletti

Hernán Vivas

The eigenvalue problem for the $p-$Laplacian

$$\boxed{-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2} u}$$

in a domain $\Omega \subset \mathbb{R}^n$ + B.C.

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in a domain $\Omega \subset \mathbb{R}^n$ + B.C.

This problem has been widely studied

- Fucik-Necas, Amann 70's
- Lindqvist 90's
- Anane-Tsouli 00's
- Cuesta-De Figueiredo-Gossez 98
- Drabek 10's
- etc....

Introduction

Main problem: Study the structure of the eigenset

$$\Sigma_p = \{\lambda \in \mathbb{R}\colon \text{is an eigenvalue of } \Delta_p\}$$

and associated eigenfunctions.

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$$\Sigma_p = \{\lambda \in \mathbb{R}\colon \text{is an eigenvalue of } \Delta_p\}$$

and associated eigenfunctions.
Some known properties:

- $\Sigma_p \subset (0, \infty)$ is closed.
- $\lambda_1 = \inf \Sigma_p$ is also the minimizer of the Rayleigh quotient $\|\nabla u\|_p^p / \|u\|_p^p$ and is isolated in Σ_p.
- There exists a sequence $\Sigma_p^{\text{var}} = \{\lambda_k\}_{k\in\mathbb{N}} \subset \Sigma_p$ (the *Ljusternik-Schnirelman eigenvalues*).
- Eigenfunctions are $C^{1,\alpha}$ for some $\alpha > 0$. Moreover, $u_1 > 0$ and is *simple*.
- much more...

The generalized Laplacian

What happens if in the $p-$Laplacian we change the *power rule* in the diffusion coefficient for a more general rule

$$\Delta_p u = \operatorname{div}(\underbrace{|\nabla u|^{p-2}}_{\text{diffusion coefficient}} \nabla u) \longleftrightarrow \Delta_\phi u = \operatorname{div}\left(\frac{\phi(|\nabla u|)}{|\nabla u|}\nabla u\right)$$

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Allows for different behaviors of the diffusion coefficient when $|\nabla u| \ll 1$ and $|\nabla u| \gg 1$.

The generalized Laplacian

Examples: Is better to state the hypotheses and examples for the primitive of ϕ

$$\Phi(t) = \int_0^t \phi(\tau)\, d\tau$$

- $\Phi(t) \sim t^p$ (i.e. the $p-$Laplacian like case)
- $\Phi(t) = \frac{t^p}{p} + \frac{t^q}{q} \longrightarrow$ the $(p,q)-$Laplacian.
- $\Phi(t) \sim t^p \ln^\alpha t$ for $t \gg 1$.
- $\Phi(t) \sim t^p \ln^\alpha t \ln^\beta(\ln t)$ for $t \gg 1$.
- $\Phi(t) = e^t - \sum_{k=0}^{n-1} \frac{t^k}{k!}$
- $\Phi(t) \sim e^{-t^{-\alpha}}$ for $t \ll 1$.

The generalized Laplacian

Elliptic problems involving the generalized Laplacian were analyzed since the 70's (e.g. Gossez '74, Lieberman '90, Tienari '00...)

$$-\Delta_\phi u = -\operatorname{div}\left(\frac{\phi(|\nabla u|)}{|\nabla u|}\nabla u\right) = f$$

in $\Omega \subset \mathbb{R}^n$ + B.C.

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in $\Omega \subset \mathbb{R}^n$ + B.C.
The source term f may depend on $x, u, \nabla u$.
A key feature is that the generalized Laplacian is the E-L equation of the energy

$$J_\Phi(u) = \int_\Omega \Phi(|\nabla u|)\, dx$$

The generalized Laplacian

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- Gossez '74 showed how to treat the problem without imposing Δ_2 condition on Φ or $\bar{\Phi}$.
- Tienari '00 extend Gossez results to deal with the eigenvalue problem

Lieberman '90 (On the natural generalizations of the natural conditions of Ladyzhenskaya and Uraltseva) proved Hölder continuity under the stronger assumption

$$\boxed{0 < \delta \le \frac{t\phi'(t)}{\phi(t)} \le g_0.} \qquad \text{(Lieberman conditions)}$$

Regularity results for the generalized Laplacian

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This condition implies the Δ_2 condition on Φ and $\bar{\Phi}$.

What is known in the fractional world?

The fractional $p-$Laplacian

$$(-\Delta_p)^s u = \text{p.v.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy$$

The eigenvalue problem:

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

- Lindgrem-Lindqvist '14
- Franzina-Palatucci '14
- Brasco-Parini-Squassina '16
- FB-Silva-Spedaletti '21

Fractional $p-$eigenvalues

The spectrum is defined as

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Similar questions to the ones posed for Σ_p are posed and (almost) answered for the fractional case.
Also the question $\Sigma_p^s \to \Sigma_p$ as $s \uparrow 1$ is considered in the above mentioned papers.

The fractional $p-$Laplacian

The operator $(-\Delta_p)^s u$ is the derivative of the fractional $p-$energy

$$J_p^s(u) = \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy.$$

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Observe that this energy can be rewritten as

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This is the $p-$energy of the Hölder quotient

$$D^s u(x, y) = \frac{u(x) - u(y)}{|x - y|^s}$$

with respect to the measure

$$d\nu_n = \frac{dxdy}{|x - y|^n}.$$

The fractional $\Phi-$energy

So, the fractional $p-$energy is written as

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This energy gives a natural way to define the fractional order Orlicz-Sobolev spaces (see also Cianchi and coauthors '21-'22)

The fractional generalized Laplacian

We defined the fractional generalized Laplacian as the derivative of the fractional $\Phi-$energy.

$$(-\Delta_\phi)^s u(x) := \text{p.v.} \int_{\mathbb{R}^n} \phi(|D^s u|) \frac{D^s u}{|D^s u|} \frac{dy}{|x-y|^{n+s}}$$

and consider problems of the form

$$\begin{cases} (-\Delta_\phi)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

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Let us begin with regularity properties for solutions

Regularity issues

Known regularity results

Regularity results for fractional *elliptic type* operators:

- Caffarelli – Silvestre → general theory for interior regularity for uniformly elliptic, fully nonlinear operators (2009–2011).
- Ros Otton – Serra → boundary regularity for uniformly elliptic fully nonlinear operators (2016).
- Di Castro – Kuusi – Palatucci → weak Harnack inequality and interior Hölder regularity for the fractional $p-$Laplacian (non uniformly elliptic) (2016)
- Iannizzotto – Mosconi – Squassina → interior and up to the boundary regularity for the fractional $p-$Laplacian (2016).

Regularity results for the fractional generalized Laplacian

Theorem (FB, Salort, Vivas '22)

Let u be a weak solution of

$$\begin{cases} (-\Delta_\phi)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Ω bounded, open with $C^{1,1}$ boundary, $f \in L^\infty(\Omega)$ and ϕ satisfies

$$0 < \delta \leq \frac{t\phi'(t)}{\phi(t)} \leq g_0, \quad t > 0.$$

Then, there exist $\alpha \in (0, s]$ and $C_0 > 0$ depending on $n, s, \Omega, \delta, g_0$ such that $u \in C^\alpha(\overline{\Omega})$ and

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C_0.$$

Weak Harnack $\Rightarrow$ local regularity

Weak Harnack ⇒ local regularity

Maximum principle detected 👍

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Maximum principle detected 👍

Weak Harnack inequality (FB, Salort, Vivas '22)

If u satisfies weakly

$$\begin{cases} (-\Delta_\phi)^s u \geq -K & \text{in } B_{R/3} \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

for some $K \geq 0$, then there exists universal $\sigma \in (0,1)$, and an explicit constant $C_0 > 0$ such that

$$\inf_{B_{R/4}} u \geq \sigma R^s \phi^{-1}\left(\fint_{B_R \setminus B_{R/2}} \phi(R^{-s}|u|)\, dx \right) - R^s \phi^{-1}(C_0 K).$$

Interior regularity follows by a standard oscillation decay argument.

Barrier ⇒ boundary regularity

Barrier $\Rightarrow$ boundary regularity

Maximum principle detected 👍

Barrier ⇒ boundary regularity

Maximum principle detected 👍

Boundary behavior (FB, Salort, Vivas '22)

Let u be a weak solution of

$$|(-\Delta_\phi)^s u| \leq K \quad \text{in } \Omega$$

for some $K > 0$.
Then

$$|u| \leq C d^s(x) \quad \text{a.e. in } \Omega$$

$d(x) = \text{dist}(x, \partial\Omega)$ and C is a positive constant depending only on s, n, δ, g_0, K and Ω.

Basic observation: $(-\Delta_\phi)^s x_+^s = 0$. This implies $(-\Delta_\phi)^s d^s$ is bounded.

The eigenvalue problem

The eigenvalue problem for $(-\Delta_\phi)^s$

The associated eigenvalue problem is the following:

$$\begin{cases} (-\Delta_\phi)^s u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

where $g \sim \phi$.

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where $g \sim \phi$.
In order to find eigenvalues, one seek for critical values of

$$\iint_{\mathbb{R}^{2n}} \Phi(|D^s u|)\, d\nu_n$$

restricted to

$$\int_\Omega \Phi(|u|)\, dx = \mu,$$

where $\mu > 0$ is a normalizing constant.

Existence of eigenvalues

The main difference is that the *normalizing constant* μ is relevant.

Existence of eigenvalues (FB, Spedaletti '23)

For any $\mu > 0$, there exists a sequence of eigenvalues $0 < \lambda_1^\mu \leq \lambda_2^\mu \leq \cdots \leq \lambda_k^\mu \uparrow \infty$

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Observe that since there is no homogeneity, the sequence depends on the normalizing constant μ.
Since we are no assuming the Δ_2 condition, the energy functional is not differentiable, so we use a Galerkin-type approximation. Moreover, since the underlying space is not reflexive we need to use the concept of complementary pair (Gossez '74)

Dependance on μ

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Theorem (FB-Salort '24)

Let

$$E(\mu) = \inf \left\{ \iint_{\mathbb{R}^{2n}} \Phi(|D^s u|)\, d\nu_n \colon \int_\Omega \Phi(|u|)\, dx = \mu \right\}.$$

Then $E(\mu)$ is differentiable and $E'(\mu) = \lambda_1^\mu$.
Moreover,

- *if Φ does not satisfies Δ_2, then $\lambda_1^\mu \to 0$ as $\mu \to \infty$*
- *if Φ does satisfies Δ_2, then $\lambda_1^\mu \to \lambda_{1,p}$ as $\mu \to \infty$ (p being the "power at infinity of Φ")*

Regularity of eigenfunctions

What about the regularity of eigenfunctions?

$$\begin{cases} (-\Delta_\phi)^s u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By the previous results, it is enough to show that u is bounded to get Hölder continuity.

Regularity of eigenfunctions

Theorem (FB, Salort, Vivas '23)

Let Φ *be a Young function satisfying*

$$1 < p^- \leq \frac{t\phi(t)}{\Phi(t)} \leq p^+$$

with $sp^+ < n$ *and* $p^+ \leq \frac{np^-}{n-sp^-}$.
There for $\mu > 0$ *there exists a constant* $C_0 = C_0(n, s, p^\pm, \mu) > 0$ *such that if* u *is an eigenfunction normalized as* $\int_\Omega \Phi(|u|)\, dx = \mu$ *then*

$$\|u\|_{L^\infty(\Omega)} \leq C_0.$$

Regularity of eigenfunctions

The proof follows a De Giorgi-type scheme: if

$$\iint_{\mathbb{R}^{2n}} \Phi(D_s u)\, d\nu_n \le c\lambda \int_\Omega \Phi(u)\, dx$$

for some constant $c = c(p^+, p^-)$, then there exists $\varepsilon_0 > 0$ such that

$$\int_\Omega \Phi(u)\, dx = \mu \le \varepsilon_0 \quad \Rightarrow \quad \|u\|_{L^\infty(\Omega)} \le 1.$$

Buena Copa América

Good Euro Cup

Thank you!!!