Hessian regularity in Hölder spaces for a semi-linear bi-Laplacian equation

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Outline

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- ⟶ Our problem;
- ⟶ Strategy;
- ⟶ Main result;
- ⟶ Sketch of the proof;
- ⟶ A simple consequence.

Our problem

Bi-Laplacian

Our interest is

Examine the regularity of weak solutions to the semi-linear bi-Laplacian equation

$$\Delta^2 u = f(x, u, Du) \text{ in } \Omega$$

where $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, and the right-hand side is a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying a polynomial growth condition.

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where $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, and the right-hand side is a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying a polynomial growth condition. That is,

$$|f(x, r, p)| \leq h(x) + C\left(|r|^\alpha + |p|^\beta\right).$$

where $h \in L^d(\Omega)$ and $\alpha, \beta \in [1, 2)$.

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This strategy is inspired by ideas introduced in the works of L. C. Evans [2003; 2009] and D. A. Gomes and H. S. Morgado [2014].

Auxiliary results

Sobolev regularity for very weak solutions

Theorem (Sobolev regularity for very weak solutions)

Fix $1 < s < \infty$. *Let* $w \in L^1_{loc}(\Omega)$ *be a very weak solution to*

$$\Delta w = g \quad in \quad \Omega,$$

with $g \in L^s_{loc}(\Omega)$, *then* $Dw \in W^{1,s}_{loc}(\Omega)$. *If* $w \in L^s_{loc}(\Omega)$, *then* $w \in W^{2,s}_{loc}(\Omega)$. *Moreover, for* $\Omega'' \Subset \Omega' \Subset \Omega$, *there exists* $C > 0$ *such that*

$$\|w\|_{W^{2,s}(\Omega'')} \leq C \left(\|w\|_{L^s(\Omega)} + \|g\|_{L^s(\Omega)} \right).$$

Lemma

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Let $u \in W^{2,q}(\Omega)$ *be a local weak solution to*

$$\Delta^2 u = f(x, u, Du) \quad in \quad \Omega,$$

with $q \geq 2$*. Then, there exists* $m \in L^q(\Omega)$ *such that* (u, m) *is a solution to*

$$\begin{cases} \Delta u = m & in\ \Omega, \\ \\ \Delta m = f(x, u, Du) & in\ \Omega. \end{cases}$$

Main result

Assumptions

For better presentation, we set $\Omega \equiv B_1$.

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$$|f(x,r,p)| \leq h(x) + C\left(|r|^\alpha + |p|^\beta\right),$$

where $h \in L^d(B_1)$, and fixed constants $C > 0$ and

$$\alpha, \beta \in [1,2).$$

Suppose further that

$$\max(\alpha,\beta)\frac{d}{2} < q \leq d.$$

Regularity estimates

Theorem (A.-Pimentel-Urbano, 2024)

Let $2 \leq q \leq d$ *and* $u \in W^{2,q}(B_1)$ *be a local weak solution to*

$$\Delta^2 u = f(x, u, Du) \quad \text{in } B_1. \tag{1}$$

Then $u \in C^{2,\sigma}_{\mathrm{loc}}(B_1)$ *for*

$$\sigma := 2 - \frac{d \max(\alpha, \beta)}{q} \in (0, 1).$$

Moreover, there exists $C > 0$ *such that*

$$\|u\|_{C^{2,\sigma}(B_{7/8})} \leq C \left(\|h\|_{L^d(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)} \right).$$

Proof

Set

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so by our assumptions we have $s \in (\frac{d}{2}, d]$. Moreover, for $B_{9/10} \Subset B_1$ it follows from the growth condition that

$$\|f(\cdot, u, Du)\|_{L^s(B_{9/10})} \leq C \left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)}^{\max(\alpha,\beta)} \right).$$

Proof

Using our approach, we can apply the Lemma, which ensures the existence of a function $m \in L^q(B_1)$ such that m is a very weak solution to the Poisson equation

$$\Delta m = f(x, u, Du) \quad \text{in} \quad B_1.$$

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Duo to Proposition, we conclude $Dm \in W^{1,s}_{loc}(B_{99/100})$. Since $d/2 < s \leq q$, we also have $m \in L^s(B_1)$ and therefore $m \in W^{2,s}_{loc}(B_{99/100})$.

Proof

Moreover, there exists $C > 0$ such that

$$\|m\|_{W^{2,s}(B_{9/10})} \leq C \left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)} + \|u\|_{W^{2,q}(B_1)}^{\max(\alpha,\beta)} \right).$$

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Moreover, there exists $C > 0$ such that

$$\|m\|_{W^{2,s}(B_{9/10})} \leq C\left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)} + \|u\|^{\max(\alpha,\beta)}_{W^{2,q}(B_1)}\right).$$

Because of Gagliardo-Nirenberg-Sobolev's embedding theorem, we obtain $m \in C^{0,\sigma}(\overline{B_{8/9}})$, with

$$\sigma := 2 - \frac{d\max(\alpha,\beta)}{q}.$$

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Because u is an L^q-strong solution to this problem, we have $u \in C^{2,\sigma}(\overline{B_{8/9}})$.

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Now we focus on the Poisson equation

$$\Delta u = m \quad \text{in} \quad B_1.$$

Because u is an L^q-strong solution to this problem, we have $u \in C^{2,\sigma}(\overline{B_{8/9}})$. Also, by Schauder's theory, there exists a positive constant, such that

$$\|u\|_{C^{2,\sigma}(B_{7/8})} \leq C\left(\|u\|_{L^\infty(B_{8/9})} + \|m\|_{C^{0,\sigma}(B_{8/9})}\right).$$

Proof

To complete the proof, we only need to notice that

$$\begin{aligned} \|m\|_{C^{0,\sigma}(B_{8/9})} &\leq C\|m\|_{W^{2,s}(B_{8/9})} \\ &\leq C\|f(\cdot,u,Du)\|_{L^s(B_{9/10})} \\ &\leq C\left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)}^{\max(\alpha,\beta)}\right). \end{aligned}$$

A simple consequence - C^∞-regularity estimates

Corollary (A.-Pimentel-Urbano, 2024)

Let $u \in W^{2,q}(B_1)$ be a weak solution to the (1), with $q \geq 2$. Suppose that our assumptions are in force, with

$$f(x,r,p) := h(x) + a(x)r + c(x) \cdot p,$$

where $h, a \in C^\infty(B_1)$ and $c \in C^\infty(B_1, \mathbb{R}^d)$. Suppose further there exists $C > 0$ such that

$$\|h\|_{C^\infty(B_1)} + \|a\|_{C^\infty(B_1)} + \|c\|_{C^\infty(B_1,\mathbb{R}^d)} \leq C.$$

Then $u \in C^\infty_{loc}(B_1)$. Moreover, for every $k \in \mathbb{N}$ and every multi-index α with $|\alpha| = k$, we have

$$\sup_{B_{7/6}} |D^\alpha u| \leq C \left(1 + \|u\|_{W^{2,q}(B_1)}\right).$$

Pre-print

Thank you for your attention!