Sharp regularity for the obstacle problem for quasilinear elliptic equations

João Vitor da Silva - jdasilva@unicamp.br

Mostly Maximum Principle (MMP 2024) – PUC-Rio - Brazil

5th edition: in Latin America for the first time

June 24th to 28th, 2024

UNICAMP

Summary


Mostly
Maximum
Principle
5th edition: in Latin America
for the first time
June 24-28, 2024
Centro Técnico Científico
Pontifícia Universidade Católica
do Rio de Janeiro (PUC-Rio)
Rua Marquês de São Vicente, 225, Gávea
Rio de Janeiro – Brazil
Organizing Committee
Isabeau Birindelli
La Sapienza, Italy
Giulio Galise
La Sapienza, Italy
Gabrielle Nornberg
University of Chile / CMM, Chile
Boyan Sirakov
PUC-Rio, Brazil
Philippe Souplet
Université Sorbonne Paris Nord, France
https://go.cmm.uchile.cl/mmp2024
PUC
Matemática PUC-Rio
SAPIENZA
SORBONNE PARIS NORD
CNPq
cnrs
CMM
CAPES
FAPERJ

The results in this Lecture are based on joint works with

Figure: Elzon C. Bezerra Júnior (UFCA-Brazil) and Romário T. Frias (UNICAMP-Brazil)

The obstacle problem: State-of-the-Art and our motivation

The obstacle problem: Old and New

- The classical obstacle problem

[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$.

[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

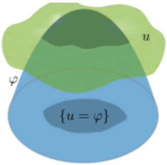
Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$. Such an equilibrium position will be the one which minimizes the potential energy involved in such a process[1].

[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$. Such an equilibrium position will be the one which minimizes the potential energy involved in such a process[1].

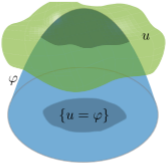


[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$. Such an equilibrium position will be the one which minimizes the potential energy involved in such a process[1].



✓ The region $\{u > \varphi\}$ is denoted the non-contact set;

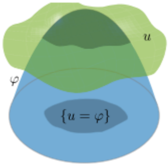
[1] For complete surveys on this topic, see

Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros – Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$. Such an equilibrium position will be the one which minimizes the potential energy involved in such a process[1].



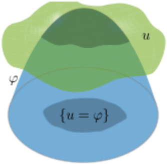
✓ The region $\{u > \varphi\}$ is denoted the non-contact set;
✓ The region $\{u = \varphi\}$ is denoted the contact set;

[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- The classical obstacle problem

Physically, the obstacle problem consists of finding the equilibrium position of an elastic membrane, which can be thought as the graph of a function $x \mapsto u(x)$ on a regular domain $\Omega \subset \mathbb{R}^n$, with a fixed boundary condition $u(x) = g(x)$ and subjected to a transversal force $f(x)$ for $x \in \Omega$. Such an equilibrium position will be the one which minimizes the potential energy involved in such a process[1].



✓ The region $\{u > \varphi\}$ is denoted the non-contact set;
✓ The region $\{u = \varphi\}$ is denoted the contact set;
✓ The region $\Gamma = \partial\{u > \varphi\}$ is denoted the free boundary.

[1] For complete surveys on this topic, see
Figalli, A., Regularity of interfaces in phase transitions via obstacle problems - Fields Medal lecture. **Proceedings of the International Congress of Mathematicians** - Rio de Janeiro 2018. Vol. I. Plenary lectures, 225-247, 2018.
Danielli, D., An overview of the obstacle problem. **Notices Amer. Math. Soc.** 67 (2020), no. 10, 1487-1497.
Ros − Oton, X., Obstacle problems and free boundaries: an overview, **SeMA J.** 75 (2018), 399-419.

The obstacle problem: Old and New

- Mathematical formulation via minimization:

$$\min_{v\in\mathcal{K}}\left(\int_{\Omega}\frac{1}{2}|\nabla v|^{2}dx+\int_{\Omega}fvdx\right) \text{ with } \mathcal{K}=\{v\in H^{1}(\Omega): v\geq\varphi \quad \text{in} \quad \Omega \text{ and } v=g \text{ on } \partial\Omega\} \quad \text{(co}$$

The obstacle problem: Old and New

- Mathematical formulation via minimization:

$$\min_{v\in\mathcal{K}}\left(\int_\Omega \frac{1}{2}|\nabla v|^2 dx + \int_\Omega f v dx\right) \text{ with } \mathcal{K} = \{v \in H^1(\Omega) : v \geq \varphi \quad \text{in} \quad \Omega \text{ and } v = g \text{ on } \partial\Omega\} \quad \text{(co}$$

Reformulating via Euler-Lagrange equation: Minimizers satisfy

$$\begin{cases} \Delta u & \leq & f & \text{in } \Omega \ \text{(in the weak sense)} \\ \Delta u & = & f & \text{in } \{u > \varphi\} \ \text{(in the weak sense)} \\ u & \geq & \varphi & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega \end{cases}$$

The obstacle problem: Old and New

- Mathematical formulation via minimization:

$$\min_{v\in\mathcal{K}}\left(\int_\Omega \frac{1}{2}|\nabla v|^2 dx + \int_\Omega f v dx\right) \text{ with } \mathcal{K} = \{v \in H^1(\Omega) : v \geq \varphi \text{ in } \Omega \text{ and } v = g \text{ on } \partial\Omega\} \text{ (co}$$

Reformulating via Euler-Lagrange equation: Minimizers satisfy

$$\begin{cases} \Delta u & \leq & f & \text{in } \Omega \text{ (in the weak sense)} \\ \Delta u & = & f & \text{in } \{u > \varphi\} \text{ (in the weak sense)} \\ u & \geq & \varphi & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega \end{cases}$$

[1] Wolanski, N. *Introducción a los problemas de frontera libre*. Cursos y Seminarios de Matemática - Serie B. Fascículo 2. 2007 Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires.

The obstacle problem: Old and New

- Mathematical formulation via minimization:

$$\min_{v\in\mathcal{K}}\left(\int_\Omega \frac{1}{2}|\nabla v|^2dx + \int_\Omega fvdx\right) \text{ with } \mathcal{K} = \{v \in H^1(\Omega) : v \geq \varphi \text{ in } \Omega \text{ and } v = g \text{ on } \partial\Omega\} \text{ (co}$$

Reformulating via Euler-Lagrange equation: Minimizers satisfy

$$\begin{cases} \Delta u & \leq & f & \text{in } \Omega \text{ (in the weak sense)} \\ \Delta u & = & f & \text{in } \{u > \varphi\} \text{ (in the weak sense)} \\ u & \geq & \varphi & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega \end{cases}$$

[1] Wolanski, N. *Introducción a los problemas de frontera libre*. Cursos y Seminarios de Matemática - Serie B. Fascículo 2. 2007 Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires.

Moreover:

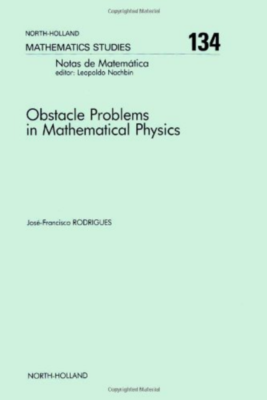
1. Existence/uniqueness of minimizers
2. H^1 regularity (Lions-Stampacchia Theorem):

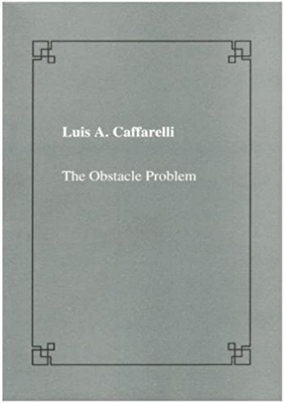
$$\|u\|_{H^1(\Omega)} \leq \mathrm{C}(\texttt{universal}).\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}\right).$$

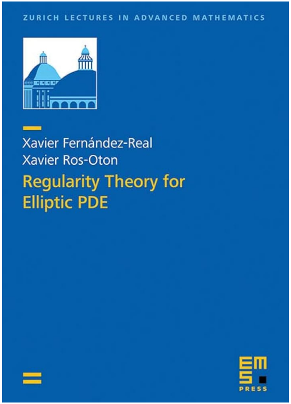
3. **Optimal regularity**: If $f \in L^\infty$ and $\varphi \in C^{1,1}$, then $u \in C^{1,1}_{\text{loc}}$ (Freshe (1972) and Brézis-Kinderleher (1974))

The obstacle problem: Old and New - References

- Rodrigues, José Francisco. **Obstacle problems in mathematical physics**. North-Holland Mathematics Studies, 134. Notas de Matemática [Mathematical Notes], 114. North-Holland Publishing Co., Amsterdam, 1987. xvi+352 pp. ISBN: 0-444-70187-7.
- Caffarelli, Luis A. **The obstacle problem**. Lezioni Fermiane. [Fermi Lectures] Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998. ii+54 pp.
- Fernández-Real, Xavier and Ros-Oton, Xavier. *Regularity theory for elliptic PDE*. Zur. Lect. Adv. Math., 28 EMS Press, Berlin, [2022], ©2022. viii+228 pp. ISBN:978-3-98547-028-0







- Quasi-linear scenario:

Reference

Andersson, J., Lindgren, E. and Shahgholian, H., *Optimal regularity for the obstacle problem for the* $p-$*Laplacian*. **J. Differential Equations** 259 (2015), no. 6, 2167-2179.

- Quasi-linear scenario:

Reference

Andersson, J., Lindgren, E. and Shahgholian, H., *Optimal regularity for the obstacle problem for the $p-$Laplacian*. **J. Differential Equations** 259 (2015), no. 6, 2167-2179.

$$\begin{cases} \text{Find } u \in \mathcal{K}_p \text{ such that} \\ \mathcal{J}_p(u) = \inf\limits_{w \in \mathcal{K}_p} \mathcal{J}_p(w) \end{cases} \tag{1}$$

where

$$\mathcal{J}_p(w) = \int_{B_1} \left(\frac{1}{p} |\nabla w|^p + f w \right) dx, \quad w \in W_0^{1,p}(B_1),$$

and

$$\mathcal{K}_p := \left\{ w \in W^{1,p}(B_1) : w \geq \phi \text{ and } w = g \text{ on } \partial B_1 \right\} \quad \text{(admissible functions)},$$

and $B_1 \subset \mathbb{R}^n$ with $n \geq 2$, $\phi \in C^{1,\beta}(\Omega)$, $f \in L^\infty(B_1)$ and $g \in W^{1,p}(B_1)$.

- Quasi-linear scenario:

Reference

Andersson, J., Lindgren, E. and Shahgholian, H., *Optimal regularity for the obstacle problem for the* $p-$*Laplacian*. **J. Differential Equations** 259 (2015), no. 6, 2167-2179.

$$\begin{cases} \text{Find } u \in \mathcal{K}_p \text{ such that} \\ \mathcal{J}_p(u) = \inf\limits_{w \in \mathcal{K}_p} \mathcal{J}_p(w) \end{cases} \tag{1}$$

where

$$\mathcal{J}_p(w) = \int_{B_1} \left(\frac{1}{p} |\nabla w|^p + f w \right) dx, \quad w \in W_0^{1,p}(B_1),$$

and

$$\mathcal{K}_p := \left\{ w \in W^{1,p}(B_1) : w \geq \phi \text{ and } w = g \text{ on } \partial B_1 \right\} \quad \text{(admissible functions)},$$

and $B_1 \subset \mathbb{R}^n$ with $n \geq 2$, $\phi \in C^{1,\beta}(\Omega)$, $f \in L^\infty(B_1)$ and $g \in W^{1,p}(B_1)$.

This statement is equivalent to find a function u such that

$$\begin{cases} \Delta_p u & = & f(x) & \text{in} & \{u > \varphi\} \cap B_1 \\ \Delta_p u & \leq & f(x) & \text{in} & B_1 \\ u(x) & \geq & \phi(x) & \text{in} & B_1 \\ u(x) & = & g(x) & \text{on} & \partial B_1. \end{cases}$$

They address the following optimal regularity estimate[2]:

Theorem (Andersson-Lindgren-Shahgholian'2015)

Let $p \in (1, \infty), \beta \in (0, 1]$, and let u be a weak solution to the p-obstacle problem in B_1 with obstacle $\phi \in C^{1,\beta}(B_1)$ and $f \in L^\infty(B_1)$. Suppose further

$$\|\phi\|_{C^{1,\beta}(B_1)} \leq \mathrm{N} \quad \textit{and} \qquad \|f\|_{L^\infty(B_1)} \leq \mathrm{L}.$$

[2] For other regularity results in the quasi-linear setting, see:
Choe and **Lewis** - **SIAM J. Math. Anal**. (1991)
Rodrigues - **Calc. Var. Partial Differential Equations** (2005)
Byun, **Cho** and **Ok** - **Forum Math**. (2016)
Eleuteri and **Passarelli di Napoli** - **Calc. Var. Partial Differential Equations** (2018)

They address the following optimal regularity estimate[2]:

Theorem (Andersson-Lindgren-Shahgholian'2015)

Let $p \in (1, \infty), \beta \in (0, 1]$, *and let* u *be a weak solution to the* p*-obstacle problem in* B_1 *with obstacle* $\phi \in C^{1,\beta}(B_1)$ *and* $f \in L^{\infty}(B_1)$. *Suppose further*

$$\|\phi\|_{C^{1,\beta}(B_1)} \leq \mathrm{N} \quad \textit{and} \quad \|f\|_{L^{\infty}(B_1)} \leq \mathrm{L}.$$

Then, for any point $y \in \Gamma \cap B_{1/2}$ *and for* $r < 1/2$

$$\sup_{x \in B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq \mathrm{C} \cdot \left(\mathrm{N}^{p-1} + \mathrm{L}\right)^{\frac{1}{p-1}} r^{1+\alpha},$$

where $\mathrm{C} = \mathrm{C}(\beta, p)$ *and*

$$\alpha = \min\left\{\frac{1}{p-1}, \beta\right\}$$

In particular,

$$\sup_{x \in B_r(y)} |u(x) - \phi(x)| \leq (\mathrm{C} + 1)\left(\mathrm{N}^{p-1} + \mathrm{L}\right)^{\frac{1}{p-1}} r^{1+\alpha}.$$

[2] For other regularity results in the quasi-linear setting, see:
Choe and **Lewis** - **SIAM J. Math. Anal**. (1991)
Rodrigues - **Calc. Var. Partial Differential Equations** (2005)
Byun, **Cho** and **Ok** - **Forum Math**. (2016)
Eleuteri and **Passarelli di Napoli** - **Calc. Var. Partial Differential Equations** (2018)

In addition, they establish the following non-degeneracy result[3]:

Proposition - Non-degeneracy of solutions

Let $p \in (2, \infty)$ and let u be a weak solution to the p-obstacle problem in B_1 with obstacle $\phi \in C^2(B_1)$ with $f \equiv 0$. Suppose further that $\Delta_p \phi < 0$. Then there is a constant $\varepsilon = \varepsilon(\sup \Delta_p \phi)$ such that for any $x^0 \in \Gamma$ and $r < \operatorname{dist}\left(x^0, \partial B_1\right)$ there holds

$$\sup_{\partial B_r(x^0) \cap \{u > \phi\}} (u - \phi) \geq \varepsilon \cdot r^2.$$

[3] See, **Figalli**, **Krummel** and **Ros − Oton** - **J. Differential Equations** (2017) for a complete and crystal clear proof for $p \in (1, \infty)$;
See also, **Challal**, **Lyaghfouri**, **Rodrigues** and **Teymurazyan** - **Interfaces Free Bound.** (2014) for related results.

Mathematical Problem Statement

We will study sharp regularity estimates to obstacle problem of p-Laplacian type

$$\begin{cases} \operatorname{div}\, \mathfrak{a}(x, \nabla u) & = & f(x) & \text{in} & \{u > \varphi\} \cap B_1 \\ \operatorname{div}\, \mathfrak{a}(x, \nabla u) & \leq & f(x) & \text{in} & B_1 \\ u(x) & \geq & \varphi(x) & \text{in} & B_1 \\ u(x) & = & 0 & \text{on} & \partial B_1, \end{cases} \tag{2}$$

where $B_1 \subset \mathbb{R}^n$ with $n \geq 2$, $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{p,q}$ where

$$\mathfrak{X}_{p,q} := \left\{ v \in W^{1,p}(B_1);\ \operatorname{div}\, \mathfrak{a}(x, \nabla v) \in L^q(B_1) \right\},$$

$1 < p < \infty$, $q > n$, $q \geq \frac{p}{p-1}$, the vector field $\mathfrak{a} : B_1 \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular at second variable.

We will study sharp regularity estimates to obstacle problem of p-Laplacian type

$$\begin{cases} \operatorname{div}\, \mathfrak{a}(x,\nabla u) = f(x) & \text{in} \quad \{u>\varphi\}\cap B_1 \\ \operatorname{div}\, \mathfrak{a}(x,\nabla u) \le f(x) & \text{in} \quad B_1 \\ u(x) \ge \varphi(x) & \text{in} \quad B_1 \\ u(x) = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{2}$$

where $B_1 \subset \mathbb{R}^n$ with $n \ge 2$, $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{p,q}$ where

$$\mathfrak{X}_{p,q} := \left\{ v \in W^{1,p}(B_1);\ \operatorname{div}\, \mathfrak{a}(x,\nabla v) \in L^q(B_1) \right\},$$

$1<p<\infty$, $q>n$, $q \ge \frac{p}{p-1}$, the vector field $\mathfrak{a} : B_1 \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular at second variable.

Structural conditions: for all $x, y \in B_1$ and $\xi, \eta \in \mathbb{R}^n$, we have

$$\begin{cases} |\mathfrak{a}(x,\xi)| + |\partial_\xi \mathfrak{a}(x,\xi)||\xi| \le \Lambda |\xi|^{p-1} \\ \lambda |\xi|^{p-2}|\eta|^2 \le \langle \partial_\xi \mathfrak{a}(x,\xi)\eta, \eta\rangle \\ \dfrac{|\mathfrak{a}(x,\xi)-\mathfrak{a}(y,\xi)|}{|\xi|^{p-1}} \le \omega(|x-y|),\ \forall\ |\xi| \neq 0, \end{cases} \tag{3}$$

where $0<\lambda \le \Lambda < \infty$ and $\omega : [0,\infty) \to [0,\infty)$ is a non-decreasing function with $\omega(0)=0$.

Finally, we further assume

$$\omega \in C^{0,\sigma}(B_1) \quad \text{and} \quad f \in L^q(B_1) \quad \text{with} \quad 0 < \sigma \leq 1. \tag{4}$$

[4] For more details, see:
Rodrigues - **Calc. Var. Partial Differential Equations** (2005)

Finally, we further assume

$$\omega \in C^{0,\sigma}(B_1) \quad \text{and} \quad f \in L^q(B_1) \quad \text{with} \quad 0 < \sigma \leq 1. \tag{4}$$

The archetypal model case for (2) is the $p-$obstacle problem with continuous coefficients

$$\left\{\begin{array}{rcccl} \operatorname{div}\left(|\nabla u|^{p-2}\mathfrak{A}(x)\nabla u\right) & = & f(x) & \text{in} & \{u > \varphi\} \cap B_1 \\ \operatorname{div}\left(|\nabla u|^{p-2}\mathfrak{A}(x)\nabla u\right) & \leq & f(x) & \text{in} & B_1 \\ u(x) & \geq & \varphi(x) & \text{in} & B_1 \\ u(x) & = & 0 & \text{on} & \partial B_1, \end{array}\right.$$

where $0 < \lambda \leq \mathfrak{A}(\cdot) \leq \Lambda$ is a matrix with entries $\sigma-$Hölder continuous.

The previous archetypal model appears, for instance, when we consider the problem of an elastic membrane over a heated plane[4]:

[4] For more details, see:
Rodrigues - **Calc. Var. Partial Differential Equations** (2005)

Finally, we further assume

$$\omega \in C^{0,\sigma}(B_1) \quad \text{and} \quad f \in L^q(B_1) \quad \text{with} \quad 0 < \sigma \leq 1. \tag{4}$$

The archetypal model case for (2) is the $p-$obstacle problem with continuous coefficients

$$\begin{cases} \text{div}\left(|\nabla u|^{p-2}\mathfrak{A}(x)\nabla u\right) & = & f(x) & \text{in} & \{u > \varphi\} \cap B_1 \\ \text{div}\left(|\nabla u|^{p-2}\mathfrak{A}(x)\nabla u\right) & \leq & f(x) & \text{in} & B_1 \\ u(x) & \geq & \varphi(x) & \text{in} & B_1 \\ u(x) & = & 0 & \text{on} & \partial B_1, \end{cases}$$

where $0 < \lambda \leq \mathfrak{A}(\cdot) \leq \Lambda$ is a matrix with entries $\sigma-$Hölder continuous.

The previous archetypal model appears, for instance, when we consider the problem of an elastic membrane over a heated plane[4]: Find a pair $(u, \Theta) \in \mathcal{K}_{\varphi,g} \times H^1(\Omega)$ such that

$$\begin{cases} \int_\Omega \Upsilon(\Theta) \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla(v-u)dx & \geq & \int_\Omega f(v-u)dx, & \forall\ v \in \mathcal{K}_{g,\varphi}. \\ \kappa \int_\Omega \nabla\Theta \cdot \nabla\Psi dx + \int_\Omega \Theta\Psi dx & = & \int_\Omega \theta\chi_{\{u=\varphi\}}\Psi dx & , \quad \forall\ \Psi \in H^1(\Omega). \end{cases}$$

[4]For more details, see:
Rodrigues - **Calc. Var. Partial Differential Equations** (2005)

Main Theorems - Part I: Sharp regularity estimates

Motivated by the Anderson *et*'s work, and investigations to the fully nonlinear degenerate scenario[5], we addressed:

Theorem 1 - Sharp regularity estimates [Bezerra Júnior-Da S.-Frias'2023]

Assuming (3) and (4) with $1 < p < \infty$ and $n \geq 2$, and considering $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{p,q}$, with $\beta \in (0,1]$, there exists a unique weak solution $u \in C^{1,\alpha}_{loc}(B_1) \cap W^{1,p}(B_1)$ of (2) with

$$\alpha = \min\left\{\beta, \min\left\{\sigma, 1 - \frac{n}{q}\right\} \cdot \min\left\{1, \frac{1}{p-1}\right\}\right\}.$$

[5] **da Silva, J.V.** and **Vivas, H.**, The obstacle problem for a class of degenerate fully non- linear operators. **Rev. Mat. Iberoam**. 37 (2021), no. 5, 1991-2020.

Motivated by the Anderson *et*'s work, and investigations to the fully nonlinear degenerate scenario[5], we addressed:

Theorem 1 - Sharp regularity estimates [Bezerra Júnior-Da S.-Frias'2023]

Assuming (3) and (4) with $1 < p < \infty$ and $n \geq 2$, and considering $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{p,q}$, with $\beta \in (0,1]$, there exists a unique weak solution $u \in C^{1,\alpha}_{loc}(B_1) \cap W^{1,p}(B_1)$ of (2) with

$$\alpha = \min\left\{\beta, \min\left\{\sigma, 1 - \frac{n}{q}\right\} \cdot \min\left\{1, \frac{1}{p-1}\right\}\right\}.$$

Furthermore, we have the following regularity estimate

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq \mathrm{C}_0 \cdot \left(\|\varphi\|^{p-1}_{C^{1,\beta}(B_1)} + \|f\|_{L^q(B_1)}\right)^{\frac{1}{p-1}},$$

where $\mathrm{C}_0 = \mathrm{C}_0\left(\alpha, n, p, q, \Lambda, \lambda, \|\omega\|_{C^{0,\sigma}(B_1)}\right)$.

[5] **da Silva, J.V.** and **Vivas, H.**, The obstacle problem for a class of degenerate fully non- linear operators. **Rev. Mat. Iberoam**. 37 (2021), no. 5, 1991-2020.

- For the existence we use a strategy of penalization:

[6] **Dong** and **Zhu** - **J. Eur. Math. Soc**. (2023)
Duzaar and **Mingione** - **Calc. Var. Partial Differential Equations** (2010)
Kuusi and **Mingione** - **J. Funct. Anal**. (2012)

- For the existence we use a strategy of penalization:

$$\begin{cases} \text{div } \mathfrak{a}(x, \nabla u_\varepsilon) = \mathrm{h}^+(x)\beta_\varepsilon(u_\varepsilon - \varphi) + f(x) - \mathrm{h}^+(x) & \text{in } \Omega \\ u_\varepsilon(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$\mathrm{h}(x) = f(x) - \text{div } \mathfrak{a}(x, \nabla\varphi)$$

for $\varphi \in \mathfrak{X}_{p,q} \cap C^{1,\beta}(\Omega)$. Moreover, for each $\varepsilon \in (0,1)$ consider $\beta_\varepsilon : \mathbb{R} \to [0,1]$ a non-decreasing Lipschitz function as follows

$$\beta_\varepsilon(s) = \begin{cases} 0, & \text{if } s \le 0 \\ \frac{s}{\varepsilon} & \text{if } 0 < s \le \varepsilon \\ 1, & \text{if } s > \varepsilon. \end{cases}$$

[6] **Dong** and **Zhu** - **J. Eur. Math. Soc**. (2023)
Duzaar and **Mingione** - **Calc. Var. Partial Differential Equations** (2010)
Kuusi and **Mingione** - **J. Funct. Anal**. (2012)

- For the existence we use a strategy of penalization:

$$\begin{cases} \text{div } \mathfrak{a}(x, \nabla u_\varepsilon) & = & \mathrm{h}^+(x)\beta_\varepsilon(u_\varepsilon - \varphi) + f(x) - \mathrm{h}^+(x) & \text{in } \Omega \\ u_\varepsilon(x) & = & 0 & \text{on } \partial\Omega \end{cases}$$

where

$$\mathrm{h}(x) = f(x) - \text{div } \mathfrak{a}(x, \nabla\varphi)$$

for $\varphi \in \mathfrak{X}_{p,q} \cap C^{1,\beta}(\Omega)$. Moreover, for each $\varepsilon \in (0,1)$ consider $\beta_\varepsilon : \mathbb{R} \to [0,1]$ a non-decreasing Lipschitz function as follows

$$\beta_\varepsilon(s) = \begin{cases} 0, & \text{if } s \le 0 \\ \frac{s}{\varepsilon} & \text{if } 0 < s \le \varepsilon \\ 1, & \text{if } s > \varepsilon. \end{cases}$$

The sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in the space[6] $C^{1,\tau_0}_{loc}(\Omega)$, where $\tau_0 \in (0,1)$, i.e.,

$$\|u_\varepsilon\|_{C^{1,\tau_0}(\Omega')} \le \mathrm{C}\left(n, \lambda, \Lambda, p, q, \Omega', \|\mathfrak{a}\|_{C^{0,\sigma}(\Omega)}, \|u_\varepsilon\|_{L^\infty(\Omega)}, \|f_\varepsilon\|_{L^q(\Omega)}\right), \quad \forall\ \Omega' \subset\subset \Omega.$$

Moreover, $u_0 := \lim_{\varepsilon \to 0} u_\varepsilon$ is a solution to original problem via stability (see, [Boccardo and Murat]).

[6] **Dong** and **Zhu** - **J. Eur. Math. Soc**. (2023)
Duzaar and **Mingione** - **Calc. Var. Partial Differential Equations** (2010)
Kuusi and **Mingione** - **J. Funct. Anal**. (2012)

We need the following A.B.P. estimates[7]:

Theorem - Aleksandrov-Bakel'man-Pucci estimates [Bezerra Júnior-Da S.-Frias'2023]

Let $f \in L^q(\Omega)$ with $q > \frac{n}{p}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ be a subsolution (resp. supersolution) of

$$-\operatorname{div} \mathfrak{a}(x, \nabla u) = f(x) \quad \text{in} \quad \Omega.$$

Then, there exists a constant $\mathrm{C} > 0$ depending on p, q, n and Λ such that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \mathrm{C} \cdot \mathrm{diam}(\Omega)^{\frac{p}{p-1} - \frac{n}{q(p-1)}} \cdot \left\| f^+ \right\|_{L^q(\Omega_{\psi_1})}^{\frac{1}{p-1}}$$

$$\left(\text{resp.} \quad \inf_{\Omega} u \geqslant -\inf_{\partial\Omega} u^- - \mathrm{C} \cdot \mathrm{diam}(\Omega)^{\frac{p}{p-1} - \frac{n}{q(p-1)}} \left\| f^- \right\|_{L^q(\Omega_{\psi_2})}^{\frac{1}{p-1}}\right)$$

[7] See, **Argiolas, Charro** and **Peral** - **Arch. Ration. Mech. Anal**. (2011)
Talenti, G. Ann. Mat. Pura Appl. (1979), for related results

- For the proof of the regularity we need some important tools like Weak Harnack Inequality, Local Maximum Principle and the following regularity estimates[8].

[8]See, **Caffarelli** and **Kinderlehrer** - **J. Analyse Math**. (1980) for related results

- For the proof of the regularity we need some important tools like Weak Harnack Inequality, Local Maximum Principle and the following regularity estimates[8].

Theorem 2 - Estimates for the linear case [Bezerra Júnior-Da S.-Frias'2023]

Let $u \in H^1_0(\Omega)$ be a weak solution of (2) satisfying the conditions of Theorem 1 with $p = 2$ and $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{2,q}$. Then, $u \in C^{1,\iota_0}_{loc}(B_1)$ with

$$\iota_0 = \min\left\{\beta, \sigma, 1 - \frac{n}{q}\right\} \quad \text{for} \quad n < q \leq \infty$$

Furthermore, we have the following estimate

$$\|u\|_{C^{1,\iota_0}(B_{1/2})} \leq \mathrm{C}_0 \cdot \left(n, \lambda, \Lambda, \beta, \sigma, q, \|f\|_{L^q(\Omega)}, \|\varphi\|_{H^1(\Omega)}\right).$$

[8] See, **Caffarelli** and **Kinderlehrer** - **J. Analyse Math**. (1980) for related results

- For the proof of the regularity we need some important tools like Weak Harnack Inequality, Local Maximum Principle and the following regularity estimates[8].

Theorem 2 - Estimates for the linear case [Bezerra Júnior-Da S.-Frias'2023]

Let $u \in H^1_0(\Omega)$ be a weak solution of (2) satisfying the conditions of Theorem 1 with $p = 2$ and $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{2,q}$. Then, $u \in C^{1,\iota_0}_{loc}(B_1)$ with

$$\iota_0 = \min\left\{\beta, \sigma, 1 - \frac{n}{q}\right\} \quad \text{for} \quad n < q \leq \infty$$

Furthermore, we have the following estimate

$$\|u\|_{C^{1,\iota_0}(B_{1/2})} \leq \mathrm{C}_0 \cdot \left(n, \lambda, \Lambda, \beta, \sigma, q, \|f\|_{L^q(\Omega)}, \|\varphi\|_{H^1(\Omega)}\right).$$

- The proof is based on some ideas from [Malý and Ziemer] and [Ok, J]

[8] See, **Caffarelli** and **Kinderlehrer** - **J. Analyse Math**. (1980) for related results

- For the proof of the regularity we need some important tools like Weak Harnack Inequality, Local Maximum Principle and the following regularity estimates[8].

Theorem 2 - Estimates for the linear case [Bezerra Júnior-Da S.-Frias'2023]

Let $u \in H^1_0(\Omega)$ be a weak solution of (2) satisfying the conditions of Theorem 1 with $p = 2$ and $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{2,q}$. Then, $u \in C^{1,\iota_0}_{loc}(B_1)$ with

$$\iota_0 = \min\left\{\beta, \sigma, 1 - \frac{n}{q}\right\} \quad \text{for} \quad n < q \leq \infty$$

Furthermore, we have the following estimate

$$\|u\|_{C^{1,\iota_0}(B_{1/2})} \leq \mathrm{C}_0 \cdot \left(n, \lambda, \Lambda, \beta, \sigma, q, \|f\|_{L^q(\Omega)}, \|\varphi\|_{H^1(\Omega)}\right).$$

- The proof is based on some ideas from [Malý and Ziemer] and [Ok, J]
- Sketch of the proof: Let $B_R(x_0) \subset\subset B_1$. Consider

$$\begin{cases} -\operatorname{div}\ \mathfrak{a}(x_0, \nabla w) = -\operatorname{div}\ \mathfrak{a}(x_0, \nabla \varphi) & \text{in} \quad B_R(x_0) \\ w(x) = u(x) & \text{on} \quad \partial B_R(x_0), \end{cases}$$

$$\begin{cases} -\operatorname{div}\ \mathfrak{a}(x_0, \nabla \mathfrak{h}) = 0 & \text{in} \quad B_R(x_0) \\ \mathfrak{h}(x) = u(x) & \text{on} \quad \partial B_R(x_0), \end{cases}$$

[8] See, **Caffarelli** and **Kinderlehrer** - **J. Analyse Math**. (1980) for related results

$$\int_{B_R(x_0)} |\nabla w - \nabla u|^2 \, dx \leq \mathrm{C}R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C}R^{n+2\beta} + \mathrm{C}R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

[9] **Maly, J.** and **Ziemer, W.P.**, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp. ISBN: 0-8218-0335-2.

$$\int_{B_R(x_0)} |\nabla w - \nabla u|^2 \, dx \leq \mathrm{C}R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C}R^{n+2\beta} + \mathrm{C}R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

$$\int_{B_R(x_0)} |\nabla w - \nabla \mathfrak{h}|^2 \, dx \leq \mathrm{C}R^{n+2\beta}$$

[9] **Maly, J.** and **Ziemer, W.P.**, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp. ISBN: 0-8218-0335-2.

$$\int_{B_R(x_0)} |\nabla w - \nabla u|^2 \, dx \leq \mathrm{C} R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C} R^{n+2\beta} + \mathrm{C} R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

$$\int_{B_R(x_0)} |\nabla w - \nabla \mathfrak{h}|^2 \, dx \leq \mathrm{C} R^{n+2\beta}$$

$$\int_{B_R(x_0)} |\nabla u - \nabla \mathfrak{h}|^2 \, dx \leq \mathrm{C} R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C} R^{n+2\beta} + \mathrm{C} R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

[9] **Maly, J.** and **Ziemer, W.P.**, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp. ISBN: 0-8218-0335-2.

$$\int_{B_R(x_0)} |\nabla w - \nabla u|^2 \, dx \leq \mathrm{C}R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C}R^{n+2\beta} + \mathrm{C}R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

$$\int_{B_R(x_0)} |\nabla w - \nabla \mathfrak{h}|^2 \, dx \leq \mathrm{C}R^{n+2\beta}$$

$$\int_{B_R(x_0)} |\nabla u - \nabla \mathfrak{h}|^2 \, dx \leq \mathrm{C}R^{2\sigma} \int_{B_R(x_0)} |\nabla u|^2 \, dx + \mathrm{C}R^{n+2\beta} + \mathrm{C}R^{n+2\left(1-\frac{n}{q}\right)} \|f\|^2_{L^q(B_1)}.$$

Hence, there exists an $R_0 > 0$ such that

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \, dx \leq \mathrm{C}\left(\frac{r}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \, dx + \mathrm{C}r^{n+2\alpha}$$

for any $0 < r \leq R \leq R_0$. For $R = R_0$ and $0 < r \leq R_0$, we have

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \, dx \leq \mathrm{C}r^{n+2\alpha}, \quad \text{with} \quad \alpha = \min\left\{\beta, \sigma, 1 - \frac{n}{q}\right\} \quad \text{for} \quad n < q \leq \infty.$$

Therefore, using a Campanato Embedding Theorem[9], we obtain the desired Hölder regularity.

[9] **Maly, J.** and **Ziemer, W.P.**, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp. ISBN: 0-8218-0335-2.

Sketch of the proof of optimal regularity

:

Sketch of the proof of optimal regularity

:

- **(Normalization)**

$$\|\varphi\|_{C^{1,\beta}(B_1)} \le \frac{1}{2} \quad \text{and} \quad \|f\|_{L^q(B_1)} \le 1.$$

Sketch of the proof of optimal regularity

:

- **(Normalization)**

$$\|\varphi\|_{C^{1,\beta}(B_1)} \leq \frac{1}{2} \quad \text{and} \quad \|f\|_{L^q(B_1)} \leq 1.$$

- For $y \in \partial\{u > \varphi\} \cap B_{1/2}$ and $r < \frac{1}{2}$ we show that

$$\sup_{x \in B_r(y)} |u(x) - u(y) - (x-y) \cdot \nabla u(y)| \leq \mathrm{C} r^{1+\alpha} \tag{5}$$

where $\mathrm{C} = \mathrm{C}(\beta, p, n, q, \|\varphi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^q(B_1)}\lambda, \Lambda)$.

Sketch of the proof of optimal regularity

:

- **(Normalization)**

$$\|\varphi\|_{C^{1,\beta}(B_1)} \leq \frac{1}{2} \quad \text{and} \quad \|f\|_{L^q(B_1)} \leq 1.$$

- For $y \in \partial\{u > \varphi\} \cap B_{1/2}$ and $r < \frac{1}{2}$ we show that

$$\sup_{x \in B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq \mathrm{C} r^{1+\alpha} \tag{5}$$

where $\mathrm{C} = \mathrm{C}(\beta, p, n, q, \|\varphi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^q(B_1)}\lambda, \Lambda)$.

- **Case 1** : $|\nabla u(y)| \leq r^{\alpha}$.

Sketch of the proof of optimal regularity

:

- **(Normalization)**

$$\|\varphi\|_{C^{1,\beta}(B_1)} \leq \frac{1}{2} \quad \text{and} \quad \|f\|_{L^q(B_1)} \leq 1.$$

- For $y \in \partial\{u > \varphi\} \cap B_{1/2}$ and $r < \frac{1}{2}$ we show that

$$\sup_{x \in B_r(y)} |u(x) - u(y) - (x-y) \cdot \nabla u(y)| \leq \mathrm{C} r^{1+\alpha} \tag{5}$$

where $\mathrm{C} = \mathrm{C}(\beta, p, n, q, \|\varphi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^q(B_1)} \lambda, \Lambda)$.

- **Case 1** : $|\nabla u(y)| \leq r^{\alpha}$.

$$\tilde{\varphi}(x) = \frac{\varphi(rx+y) - \varphi(y)}{r^{1+\alpha}} \quad \text{and} \quad \tilde{u}(x) = \frac{u(rx+y) - u(y)}{r^{1+\alpha}}.$$

Sketch of the proof of optimal regularity

:

- **(Normalization)**

$$\|\varphi\|_{C^{1,\beta}(B_1)} \leq \frac{1}{2} \quad \text{and} \quad \|f\|_{L^q(B_1)} \leq 1.$$

- For $y \in \partial\{u > \varphi\} \cap B_{1/2}$ and $r < \frac{1}{2}$ we show that

$$\sup_{x \in B_r(y)} |u(x) - u(y) - (x-y)\cdot \nabla u(y)| \leq \mathrm{C} r^{1+\alpha} \tag{5}$$

where $\mathrm{C} = \mathrm{C}(\beta, p, n, q, \|\varphi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^q(B_1)}\lambda, \Lambda)$.

- **Case 1** : $|\nabla u(y)| \leq r^{\alpha}$.

$$\tilde{\varphi}(x) = \frac{\varphi(rx+y) - \varphi(y)}{r^{1+\alpha}} \quad \text{and} \quad \tilde{u}(x) = \frac{u(rx+y) - u(y)}{r^{1+\alpha}}.$$

$$\|\tilde{\varphi}\|_{L^\infty(B_1)} \quad \leq \quad \frac{r^{1+\beta}}{2r^{1+\alpha}} + 1 \leq \frac{3}{2}.$$

Then, $\tilde{u} \geq \tilde{\varphi}$, it satisfies

$$\text{div } \tilde{\mathfrak{a}}(x, \nabla \tilde{u}) \leq \tilde{f}(x) \quad \text{in } B_1$$

for

$$\tilde{\mathfrak{a}}(x,\xi) := r^{\alpha(1-p)}\mathfrak{a}(rx, r^{\alpha}\xi) \quad \text{and} \quad \tilde{f}(x) = r^{\alpha(1-p)+1} f(rx+y).$$

10

$$\text{(WHI)} \quad \|u\|_{L^{\gamma}(B_{\kappa r})} \leq \mathrm{C}\left[\inf_{B_{\tau_0 r}} u(x) + \|f\|^{\frac{1}{p-1}}_{L^q(B_1)}\right], \ \kappa, \tau_0 \in (0,1) \text{ and } \gamma \in \left(0, \frac{n(p-1)}{n-p}\right), \ u \text{ a weak super-sol..}$$

11

$$\text{(LMP)} \quad \sup_{B_{\kappa r}} u^+(x) \leq \frac{\mathrm{C}(p,n,\lambda,j,r)}{\sqrt[j]{(1-\kappa)^n}}\left[\|u^+\|_{L^j(B_r)} + \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}\right], \quad 0 < j \leq p, \quad u \text{ a weak sub-solution.}$$

Then, $\tilde{u} \geq \tilde{\varphi}$, it satisfies

$$\mathrm{div}\ \tilde{\mathfrak{a}}(x, \nabla \tilde{u}) \leq \tilde{f}(x) \quad \text{in } B_1$$

for

$$\tilde{\mathfrak{a}}(x, \xi) := r^{\alpha(1-p)} \mathfrak{a}(rx, r^{\alpha}\xi) \quad \text{and} \quad \tilde{f}(x) = r^{\alpha(1-p)+1} f(rx + y).$$

$$\begin{aligned} |\tilde{\mathfrak{a}}(x, \xi) - \tilde{\mathfrak{a}}(y, \xi)| &\leq r^{\alpha(1-p)+\sigma} \omega(|x-y|)|\xi|^{p-1} \\ &\leq \omega(|x-y|)|\xi|^{p-1}. \end{aligned}$$

10

(WHI) $\|u\|_{L^{\gamma}(B_{\kappa r})} \leq \mathrm{C}\left[\inf_{B_{\tau_0 r}} u(x) + \|f\|_{L^q(B_1)}^{\frac{1}{p-1}}\right]$, $\kappa, \tau_0 \in (0,1)$ and $\gamma \in \left(0, \frac{n(p-1)}{n-p}\right)$, u a weak super-sol..

11

(LMP) $\sup_{B_{\kappa r}} u^+(x) \leq \frac{\mathrm{C}(p, n, \lambda, j, r)}{\sqrt[j]{(1-\kappa)^n}}\left[\|u^+\|_{L^j(B_r)} + \|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}\right]$, $0 < j \leq p$, u a weak sub-solution.

Then, $\tilde{u} \geq \tilde{\varphi}$, it satisfies

$$\mathrm{div}\ \tilde{\mathfrak{a}}(x, \nabla \tilde{u}) \leq \tilde{f}(x) \quad \text{in } B_1$$

for

$$\tilde{\mathfrak{a}}(x,\xi) := r^{\alpha(1-p)}\mathfrak{a}(rx, r^{\alpha}\xi) \quad \text{and} \quad \tilde{f}(x) = r^{\alpha(1-p)+1} f(rx+y).$$

$$\begin{aligned} |\tilde{\mathfrak{a}}(x,\xi) - \tilde{\mathfrak{a}}(y,\xi)| &\leq r^{\alpha(1-p)+\sigma}\omega(|x-y|)|\xi|^{p-1} \\ &\leq \omega(|x-y|)|\xi|^{p-1}. \end{aligned}$$

$$\|\tilde{f}\|_{L^q(B_1)} = r^{\alpha(1-p)+1-n/q}\|f\|_{L^q(B_1)} \leq 1.$$

10

(WHI) $\|u\|_{L^{\gamma}(B_{\kappa r})} \leq \mathrm{C}\left[\inf_{B_{\tau_0 r}} u(x) + \|f\|_{L^q(B_1)}^{\frac{1}{p-1}}\right]$, $\kappa, \tau_0 \in (0,1)$ and $\gamma \in \left(0, \frac{n(p-1)}{n-p}\right)$, u a weak super-sol..

11

(LMP) $\sup_{B_{\kappa r}} u^+(x) \leq \frac{\mathrm{C}(p,n,\lambda,j,r)}{\sqrt[j]{(1-\kappa)^n}}\left[\|u^+\|_{L^j(B_r)} + \|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}\right]$, $0 < j \leq p$, u a weak sub-solution.

Then, $\tilde{u} \geq \tilde{\varphi}$, it satisfies

$$\operatorname{div}\ \tilde{\mathfrak{a}}(x, \nabla \tilde{u}) \leq \tilde{f}(x) \quad \text{in } B_1$$

for

$$\tilde{\mathfrak{a}}(x,\xi) := r^{\alpha(1-p)}\mathfrak{a}(rx, r^{\alpha}\xi) \quad \text{and} \quad \tilde{f}(x) = r^{\alpha(1-p)+1}f(rx+y).$$

$$\begin{aligned} |\tilde{\mathfrak{a}}(x,\xi) - \tilde{\mathfrak{a}}(y,\xi)| &\leq r^{\alpha(1-p)+\sigma}\omega(|x-y|)|\xi|^{p-1} \\ &\leq \omega(|x-y|)|\xi|^{p-1}. \end{aligned}$$

$$\|\tilde{f}\|_{L^q(B_1)} = r^{\alpha(1-p)+1-n/q}\|f\|_{L^q(B_1)} \leq 1.$$

Therefore, using the Weak Harnack Inequality[10] and the Local Maximum Principle[11], we obtain the desired regularity estimate.

10

(WHI) $\|u\|_{L^{\gamma}(B_{\kappa r})} \leq \mathrm{C}\left[\inf_{B_{\tau_0 r}} u(x) + \|f\|_{L^q(B_1)}^{\frac{1}{p-1}}\right]$, $\kappa, \tau_0 \in (0,1)$ and $\gamma \in \left(0, \dfrac{n(p-1)}{n-p}\right)$, u a weak super-sol..

11

(LMP) $\sup_{B_{\kappa r}} u^{+}(x) \leq \dfrac{\mathrm{C}(p,n,\lambda,j,r)}{\sqrt[j]{(1-\kappa)^n}}\left[\|u^{+}\|_{L^j(B_r)} + \|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}\right]$, $0 < j \leq p$, u a weak sub-solution.

- **Case 2**: $\mathfrak{L} \geq |\nabla u(y)| \geq r^{\alpha}$.

Let $r_y^{\alpha} = |\nabla u(y)|$. From Case 1, we know the following

$$\sup_{B_{r_y}(y)} |u(x) - u(y)| \leq \mathrm{C} r_y^{1+\alpha}. \tag{6}$$

- **Case 2**: $\mathfrak{L} \geq |\nabla u(y)| \geq r^{\alpha}$.

Let $r_y^{\alpha} = |\nabla u(y)|$. From Case 1, we know the following

$$\sup_{B_{r_y}(y)} |u(x) - u(y)| \leq \mathrm{C} r_y^{1+\alpha}. \tag{6}$$

Define

$$\begin{cases} \hat{\varphi}(x) &= \dfrac{\varphi(r_y x + y) - \varphi(y)}{r_y^{1+\alpha}} \\ \hat{u}(x) &= \dfrac{u(r_y x + y) - u(y)}{r_y^{1+\alpha}} \\ \hat{f}(x) &= r_y^{1-\alpha(p-1)} f(r_y x + y). \end{cases}$$

Note that

$$|\nabla \hat{\varphi}(0)| = |\nabla \hat{u}(0)| = 1. \tag{7}$$

Moreover, $\hat{u}(x) \geq \hat{\varphi}(x)$ in B_1 and satisfies

$$\mathrm{div}\ \hat{\mathfrak{a}}(x, \nabla \hat{u}) \leq \hat{f}(x) \quad \text{in } B_1$$

for

$$\hat{\mathfrak{a}}(x, \xi) := r_y^{\alpha(1-p)} \mathfrak{a}(r_y x, r_y^{\alpha} \xi) \quad \text{and} \quad \hat{f}(x) = r_y^{\alpha(1-p)+1} f(r_y x + y).$$

Moreover, $\hat{u}(x) \geq \hat{\varphi}(x)$ in B_1 and satisfies

$$\mathrm{div}\ \hat{\mathfrak{a}}(x, \nabla \hat{u}) \leq \hat{f}(x) \quad \text{in } B_1$$

for

$$\hat{\mathfrak{a}}(x, \xi) := r_y^{\alpha(1-p)} \mathfrak{a}(r_y x, r_y^{\alpha} \xi) \quad \text{and} \quad \hat{f}(x) = r_y^{\alpha(1-p)+1} f(r_y x + y).$$

Using that $\hat{u} \in C^{1,\tau_0}_{loc}(B_1)$ and $|\nabla \hat{u}(0)| = 1$ we can find a radius r_0 and a constant c_0 such that

$$\mathrm{c}_0 \leq |\nabla \hat{u}(x)| \leq \mathrm{c}_0^{-1} \quad \text{in} \quad B_{r_0}.$$

Moreover, $\hat{u}(x) \geq \hat{\varphi}(x)$ in B_1 and satisfies

$$\text{div } \hat{\mathfrak{a}}(x, \nabla \hat{u}) \leq \hat{f}(x) \quad \text{in } B_1$$

for

$$\hat{\mathfrak{a}}(x, \xi) := r_y^{\alpha(1-p)} \mathfrak{a}(r_y x, r_y^{\alpha} \xi) \quad \text{and} \quad \hat{f}(x) = r_y^{\alpha(1-p)+1} f(r_y x + y).$$

Using that $\hat{u} \in C^{1,\tau_0}_{loc}(B_1)$ and $|\nabla \hat{u}(0)| = 1$ we can find a radius r_0 and a constant c_0 such that

$$\mathrm{c}_0 \leq |\nabla \hat{u}(x)| \leq \mathrm{c}_0^{-1} \quad \text{in} \quad B_{r_0}.$$

Then, $\hat{\mathfrak{a}}(\cdot, \cdot)$ satisfies (3) (Structural Conditions) for "$p = 2$" in B_{r_0}. From the Theorem 2 (linear case), we have $\hat{u} \in C^{1,\iota_0}_{loc}(B_{r_0 r_y}(y))$.

Moreover, $\hat{u}(x) \geq \hat{\varphi}(x)$ in B_1 and satisfies

$$\mathrm{div}\ \hat{\mathfrak{a}}(x, \nabla \hat{u}) \leq \hat{f}(x) \quad \text{in } B_1$$

for

$$\hat{\mathfrak{a}}(x,\xi) := r_y^{\alpha(1-p)} \mathfrak{a}(r_y x, r_y^{\alpha} \xi) \quad \text{and} \quad \hat{f}(x) = r_y^{\alpha(1-p)+1} f(r_y x + y).$$

Using that $\hat{u} \in C^{1,\tau_0}_{loc}(B_1)$ and $|\nabla \hat{u}(0)| = 1$ we can find a radius r_0 and a constant c_0 such that

$$\mathrm{c}_0 \leq |\nabla \hat{u}(x)| \leq \mathrm{c}_0^{-1} \quad \text{in} \quad B_{r_0}.$$

Then, $\hat{\mathfrak{a}}(\cdot,\cdot)$ satisfies (3) (Structural Conditions) for "$p = 2$" in B_{r_0}. From the Theorem 2 (linear case), we have $\hat{u} \in C^{1,\iota_0}_{loc}(B_{r_0 r_y}(y))$.

Therefore, by re-scaling, we obtain the desired sharp regularity estimates

$$\sup_{B_r(y)} |u(x) - u(y) - (x-y)\cdot \nabla u(y)| \leq \mathrm{C} r^{1+\iota_0} \leq \mathrm{C} r^{1+\alpha},$$

for $r \leq r_0 r_y = r_0 |\nabla u(y)|^{\frac{1}{\alpha}}$.

Main Theorems - Part II: Non-degeneracy of solutions and beyond

For obtaining further geometric properties of solutions, we need to assume:

- $$\left|\frac{\partial \mathfrak{a}_i}{\partial x_i}(x,\xi)\right| \leq \Lambda_0 |\xi|^{p-1} \qquad \text{(for every } 1 \leq i \leq n, \text{ for } \; p \geq 2), \tag{8}$$

For obtaining further geometric properties of solutions, we need to assume:

- $$\left|\frac{\partial \mathfrak{a}_i}{\partial x_i}(x,\xi)\right| \leq \Lambda_0|\xi|^{p-1} \qquad \text{(for every } 1 \leq i \leq n, \text{ for } p \geq 2), \tag{8}$$

- $$0 < \mathrm{c}_0 := \inf_{B_1} f(x) \leqslant f(x), \qquad \text{a.e. in } B_1. \tag{9}$$

For obtaining further geometric properties of solutions, we need to assume:

- $$\left|\frac{\partial \mathfrak{a}_i}{\partial x_i}(x,\xi)\right| \leq \Lambda_0|\xi|^{p-1} \qquad \text{(for every } 1 \leq i \leq n, \text{ for } p \geq 2), \tag{8}$$

- $$0 < \mathrm{c}_0 := \inf_{B_1} f(x) \leqslant f(x), \qquad \text{a.e. in } B_1. \tag{9}$$

- We will assume that $\varphi \in C^{1,1}(B_1)$ such that

$$2^{p-2}n\Lambda_0|\nabla\varphi|^{p-1} + \Lambda \max\left\{1, 2^{p-3}\right\}|\nabla\varphi|^{p-2}\sum_{i,j=1}^{n}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right| \leq \mathrm{c}_0 - \delta, \tag{10}$$

  where $0 < \delta < \mathrm{c}_0$ is a fixed constant.

For obtaining further geometric properties of solutions, we need to assume:

- $$\left|\frac{\partial \mathfrak{a}_i}{\partial x_i}(x,\xi)\right| \le \Lambda_0|\xi|^{p-1} \qquad \text{(for every } 1 \le i \le n, \text{ for } p \ge 2), \tag{8}$$

- $$0 < \mathrm{c}_0 := \inf_{B_1} f(x) \leqslant f(x), \qquad \text{a.e. in } B_1. \tag{9}$$

- We will assume that $\varphi \in C^{1,1}(B_1)$ such that

$$2^{p-2}n\Lambda_0|\nabla\varphi|^{p-1} + \Lambda \max\left\{1, 2^{p-3}\right\}|\nabla\varphi|^{p-2}\sum_{i,j=1}^{n}\left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right| \le \mathrm{c}_0 - \delta, \tag{10}$$

where $0 < \delta < \mathrm{c}_0$ is a fixed constant.

Remark: We must stress that the assumption (10) implies the following

$$\mathrm{div}\ \mathfrak{a}(x,\nabla\varphi) < c_0$$

Therefore, under the previous assumptions, we can address the following non-degeneracy result:

Theorem 3 - Non-degeneracy of solutions [Bezerra Júnior-Da S.-Frias'2023]

Assume assumptions (8) and (9) are in force. Let $p \in (2, \infty)$ and $u \in W^{1,p}(B_1)$ be a weak solution of the obstacle problem (2) for $\varphi \in C^{1,1}(B_1)$ satisfying (10). Then, there exist $r^* > 0$ and a constant $\epsilon_0 = \epsilon_0$(universal)) such that for every $x^0 \in \overline{\{u > \varphi\}} \cap B_1$ and every $r \in (0, r^*)$ fulfilling $B_r(x^0) \subset B_1$ we obtain

$$\sup_{\partial B_r(x_0) \cap \{u > \varphi\}} (u - \varphi) \geq \epsilon_0 \cdot r^2.$$

Therefore, under the previous assumptions, we can address the following non-degeneracy result:

Theorem 3 - Non-degeneracy of solutions [Bezerra Júnior-Da S.-Frias'2023]

Assume assumptions (8) and (9) are in force. Let $p \in (2, \infty)$ and $u \in W^{1,p}(B_1)$ be a weak solution of the obstacle problem (2) for $\varphi \in C^{1,1}(B_1)$ satisfying (10). Then, there exist $r^* > 0$ and a constant $\epsilon_0 = \epsilon_0(\text{universal}))$ such that for every $x^0 \in \overline{\{u > \varphi\}} \cap B_1$ and every $r \in (0, r^*)$ fulfilling $B_r(x^0) \subset B_1$ we obtain

$$\sup_{\partial B_r(x_0) \cap \{u > \varphi\}} (u - \varphi) \geq \epsilon_0 \cdot r^2.$$

Particularly, along the set of critical points, i.e., if $|\nabla \varphi(x^0)| = 0$, then

$$\sup_{\partial B_r(x_0) \cap \{u > \varphi\}} (u - \varphi) \geq \epsilon_0 \cdot r^{1+\gamma},$$

for any $\frac{1}{p-1} \leq \gamma \leq 1$.

Sketch of the proof of Non-degeneracy

Sketch of the proof of Non-degeneracy

By continuity, it is sufficient to prove the result for $y \in \{u > \varphi\}$. Take $y \in \{u > \varphi\}$. Let

$$v(x) := \varphi(x) + \epsilon_0 |x - y|^{1+\gamma}$$

be a comparison function, where $\gamma > 0$ will be determined *a posteriori*.

Sketch of the proof of Non-degeneracy

By continuity, it is sufficient to prove the result for $y \in \{u > \varphi\}$. Take $y \in \{u > \varphi\}$. Let

$$v(x) := \varphi(x) + \epsilon_0 |x - y|^{1+\gamma}$$

be a comparison function, where $\gamma > 0$ will be determined *a posteriori*. For suitable values of γ, there exists a universal $r^* > 0$ such that:

$$\forall r \in (0, r^*), \quad \forall y \in \overline{\{u > \varphi\}}, \quad \forall x \in B_r(y) \subset B_1, \qquad \text{div}\, \mathfrak{a}(x, \nabla v) \leq \mathrm{c}_0$$

Since $u(y) > \varphi(y) = v(y)$, thus by the Comparison Principle, there must exists a $z_y \in \partial(B_r(y) \cap \{u > \varphi\})$ such that $u(z_y) \geqslant v(z_y)$. Moreover, note that:

$$\partial \left(B_r(y) \cap \{u > \varphi\}\right) = \left(\partial B_r(y) \cap \{u > \varphi\}\right) \cup \left(B_r(y) \cap \partial\{u > \varphi\}\right)$$

Since $u < v$ in $B_r(y) \cap \partial\{u > \varphi\}$, we have $z_y \notin B_r(y) \cap \partial\{u > \varphi\}$, which implies that $z_y \in \partial B_r(y) \cap \{u > \varphi\}$.

Sketch of the proof of Non-degeneracy

By continuity, it is sufficient to prove the result for $y \in \{u > \varphi\}$. Take $y \in \{u > \varphi\}$. Let

$$v(x) := \varphi(x) + \epsilon_0 |x - y|^{1+\gamma}$$

be a comparison function, where $\gamma > 0$ will be determined *a posteriori*. For suitable values of γ, there exists a universal $r^* > 0$ such that:

$$\forall r \in (0, r^*)\,, \quad \forall y \in \overline{\{u > \varphi\}}, \quad \forall x \in B_r(y) \subset B_1, \qquad \text{div}\ \mathfrak{a}(x, \nabla v) \leq \mathrm{c}_0$$

Since $u(y) > \varphi(y) = v(y)$, thus by the Comparison Principle, there must exists a $z_y \in \partial(B_r(y) \cap \{u > \varphi\})$ such that $u\,(z_y) \geqslant v\,(z_y)$. Moreover, note that:

$$\partial\,(B_r(y) \cap \{u > \varphi\}) = (\partial B_r(y) \cap \{u > \varphi\}) \cup (B_r(y) \cap \partial\{u > \varphi\})$$

Since $u < v$ in $B_r(y) \cap \partial\{u > \varphi\}$, we have $z_y \notin B_r(y) \cap \partial\{u > \varphi\}$, which implies that $z_y \in \partial B_r(y) \cap \{u > \varphi\}$.

Therefore,

$$\sup_{\partial B_r(y) \cap \{u > \varphi\}} (u - \varphi) \geq \epsilon_0 \cdot r^2.$$

and along the set of critical points,

$$\sup_{\partial B_r(y) \cap \{u > \varphi\}} (u - \varphi) \geq \epsilon_0 \cdot r^{1+\gamma} \quad \text{for any} \quad \frac{1}{p-1} \leq \gamma \leq 1.$$

Applications and beyond

As a result of Theorem 1, we obtain the following: if a solution of p-evolution obstacle problem has its time derivative L^q-integrable, then it exhibits optimal growth of $(1+\alpha_\sharp)-$order, where

$$\alpha_\sharp = \frac{1 - \frac{n+2}{q}}{(p-1)\left(1 - \frac{1}{q}\right) + \frac{1}{q}} = \frac{q - (n+2)}{(p-1)(q-1)+1}.$$

Such a growth occurs in the spatial variable along free boundary points.

As a result of Theorem 1, we obtain the following: if a solution of p-evolution obstacle problem has its time derivative L^q-integrable, then it exhibits optimal growth of $(1+\alpha_\sharp)-$order, where

$$\alpha_\sharp = \frac{1-\frac{n+2}{q}}{(p-1)\left(1-\frac{1}{q}\right)+\frac{1}{q}} = \frac{q-(n+2)}{(p-1)(q-1)+1}.$$

Such a growth occurs in the spatial variable along free boundary points.

Theorem (Bezerra Júnior-Da S.-Frias'2023)

Let $p \in (1,\infty)$, and let u be a weak solution to the inhomogeneous $p-$parabolic obstacle problem

$$\begin{cases} \max\{\operatorname{div} \mathfrak{a}(x,\nabla u) - u_t - f, u-\varphi\} & = & 0 & \textit{in} & Q_r^- \\ u & = & 0 & \textit{on} & \partial_p Q_r^- \end{cases} \tag{11}$$

with the obstacle $\varphi \in C_x^2(Q_1^-)$. Suppose further that $q > n+2$, $\sigma = 1$ and

$$\|u_t\|_{L^q(Q_1^-)} \leq \mathbf{L}_*, \quad \|f\|_{L^q(Q_1^-)} \leq \mathbf{L}_\sharp \quad \textit{and} \quad \|\varphi\|_{C_x^2(Q_1^-)} \leq \mathbf{N}_*.$$

As a result of Theorem 1, we obtain the following: if a solution of p-evolution obstacle problem has its time derivative L^q-integrable, then it exhibits optimal growth of $(1+\alpha_\sharp)-$order, where

$$\alpha_\sharp = \frac{1-\frac{n+2}{q}}{(p-1)\left(1-\frac{1}{q}\right)+\frac{1}{q}} = \frac{q-(n+2)}{(p-1)(q-1)+1}.$$

Such a growth occurs in the spatial variable along free boundary points.

Theorem (Bezerra Júnior-Da S.-Frias'2023)

Let $p \in (1,\infty)$, and let u be a weak solution to the inhomogeneous $p-$parabolic obstacle problem

$$\begin{cases} \max\left\{\operatorname{div} \mathfrak{a}(x,\nabla u) - u_t - f, u-\varphi\right\} & = & 0 & \textit{in} & Q_r^- \\ u & = & 0 & \textit{on} & \partial_p Q_r^- \end{cases} \tag{11}$$

with the obstacle $\varphi \in C^2_x(Q_1^-)$. Suppose further that $q > n+2$, $\sigma = 1$ and

$$\|u_t\|_{L^q(Q_1^-)} \le \mathbf{L}_*, \quad \|f\|_{L^q(Q_1^-)} \le \mathbf{L}_\sharp \quad \textit{and} \quad \|\varphi\|_{C^2_x(Q_1^-)} \le \mathbf{N}_*.$$

Then, for any point $(y,s) \in \partial\{u>\varphi\} \cap Q^-_{\frac{1}{2}}$ and for $r \in \left(0, \frac{1}{4}\right)$, there holds

$$\sup_{(x,t)\in Q_r^-(y,s)} |u(x,t)-u(y,s)-\nabla u(y,s)\cdot(x-y)| \le \mathrm{C}(p,\mathbf{L}_*,\mathbf{L}_\sharp,\mathbf{N}_*) r^{1+\alpha_\sharp}.$$

References

[1] Andersson, J., Lindgren, E. and Shahgholian, H., *Optimal regularity for the obstacle problem for the p-Laplacian*. **J. Differential Equations** 259 (2015), no. 6, 2167-2179.

[2] Bezerra Júnior, E.C., da Silva, J.V. and Tomilhero Frias, R. A version of the Aleksandrov-Bakel'man-Pucci Maximum Principle for p-Laplacian type equations and applications. Preprint 2023.

[3] Bezerra Júnior, E.C., da Silva, J.V. and Tomilhero Frias, R. Sharp regularity for the obstacle problem for quasilinear elliptic equations of $p-$Laplacian type and applications. Preprint 2023.

[4] Boccardo, L. and Murat, F. *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*. **Nonlinear Anal**. 19 (1992), no. 6, 581–597.

[5] Byun, S., Cho, Y., Ok, J.: Global gradient estimates for nonlinear obstacle problems with nonstandard growth. Forum Math. 28(4), 729–747 (2016).

[6] Malý, J. and Ziemer, W.P., *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp. ISBN: 0-8218-0335-2.

[7] Ok, J., *Gradient continuity for nonlinear obstacle problems*. **Mediterr. J. Math.** 14 (2017), no. 1, Paper No. 16, 24 pp.

[8] Rodrigues, J.F., *Stability remarks to the obstacle problem for p-Laplacian type equations*. **Calc. Var. Partial Differential Equations** 23 (2005), no. 1, 51–65.

Acknowledgments and Sponsors

This Work/Lecture has been financially supported by:

CNPq

FAEPEX
Fundo de Apoio ao Ensino, Pesquisa e Extensão

INCTMat
Instituto Nacional de Ciência e Tecnologia de Matemática

Thank you very much for your attention!

Welcome to visit the Mathematics Department at UNICAMP!

Do you want to know a bit about my research? Please, follow me on ResearchGate!

https://www.researchgate.net/profile/Joao-Da-Silva-13