Degenerate logistic type equations

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Mostly Maximum Principle - 5th edition

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Degenerate logistic type equations

Joint work with L. Maia.

Investigate solutions of the degenerate logistic equation with a general superlinear nonlinearity

$$\begin{cases} \partial_t u = \Delta u + \lambda u + b(x) f(u), \quad (x,t) \text{ in } \Omega \times \mathbb{R}^+, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0(x), \end{cases}$$

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- Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$
- λ is a real positive parameter,
- $b : \bar{\Omega} \mapsto \mathbb{R} \in L^\infty(\bar{\Omega})$ satisfies $b(x) \leq 0$ and $b(x) = 0$ in a connected set $\Omega_0 \subset\subset \Omega$ with positive Lebesgue measure and smooth boundary.

Main goal

- Exploit the interplay of variational methods with dynamical systems to analyze the behavior of solutions.
- Nehari manifold: the existence and non-existence of stationary solutions. It is used to locate the stationary solutions, and identify convergence regions of evolutionary trajectories.
- Detailed picture of the positive dynamics and local behavior of solutions near a nodal equilibrium.

Motivation I: mathematical model in population dynamics

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- DBC: Ω is fully surrounded by completely hostile regions
- u_0 initial population distribution

Interplay between laws of population dynamics

Heterogeneous environment

1. Spatial logistic equation $g(x,u) = u, b < 0, \Omega_- = \{b(x) < 0\} = \Omega$

$$u_t - \Delta u = \lambda u + b(x)u^2, \quad x \in \Omega$$

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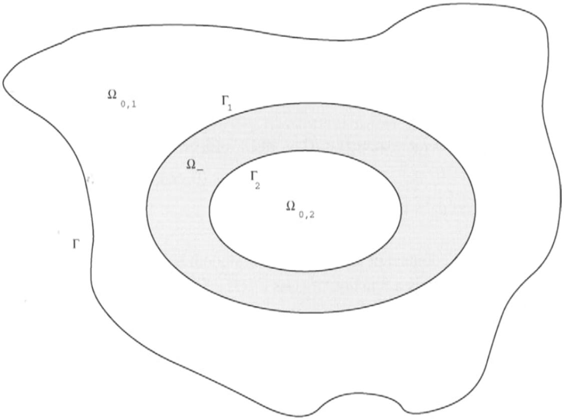
2. Spatial Malthus equation $b \equiv 0$, $\Omega_- = \emptyset, \quad x \in \Omega$

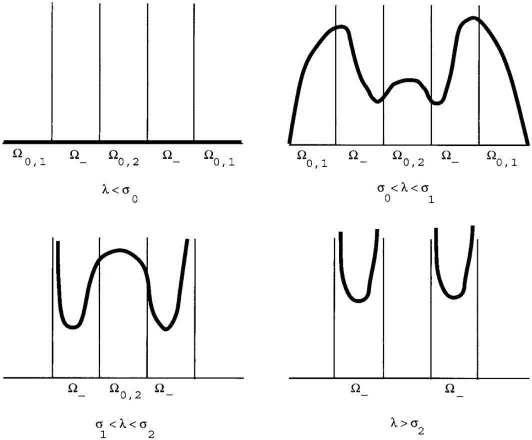
Example: nodal configuration induced by $b(x)$



López-Gómez 05: two components $\Omega_0 = \Omega_{0,1} \cup \Omega_{0,2}$ (with corresponding first eigenvalues σ_1 and σ_2).

Example: nodal configuration induced by $b(x)$

Limiting profiles of solutions



López-Gómez 05: σ_0 first eigenvalue of Ω: $\sigma_0 < \sigma_1 < \sigma_2$.

Motivation II: metasolutions

- The limiting profile of $u(x, t)$ as time $t \to \infty$ becomes infinity in the refuges.
- Metasolutions are used to describe the dynamics of classes of spatially heterogeneous semilinear parabolic problems (ecology).
- Number of technicalities involved in their study.
  - Formal concept of metasolutions: Gómez-Reñasco and López-Gómez 02.
  - Singular boundary value problems, numerical simulations: Molina-Meyer and Prieto-Medina 20
  - 1D degenerate boundary value problems, existence of nodal solutions: López-Gómez and Rabinowitz 15

Some elliptic background

$$-\Delta u = \lambda u + b(x)u^{p-1}u, \qquad x \in \Omega,$$

- $b(x)$ changes sign
  - Ouyang 92: sub and super solution
  - Alama and Tarantello 93 : bifurcation theory, variational methods
  - Brown and Zhang 03: Nehari manifold

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  - Brown and Zhang 03: Nehari manifold
- $b(x) \leq 0$
  - del Pino and Felmer 95: variational methods (linking) for analysis of multiple solutions

Some parabolic background

$$u_t - \Delta u = \lambda u + b(x)g(x,u)u, \qquad (x,t) \text{ in } \Omega \times \mathbb{R}^+$$

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- Stability of positive and negative stationary solutions
  - Kajikiya 12

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- $b(x) > 0$ with power nonlinearity, or hyperbolic equation
  - Gazzola and Weth 05, Sattinger 75

Some parabolic background: asymptoticaly linear

Problems in saturable media

$$u_t - \Delta u = \lambda u + f(u), \qquad (x,t) \text{ in } \Omega \times \mathbb{R}^+$$

Some parabolic background: asymptoticaly linear

Problems in saturable media

$$u_t - \Delta u = \lambda u + f(u), \qquad (x,t) \text{ in } \Omega \times \mathbb{R}^+$$

(f_1) $f \in C^2(\mathbb{R})$, f odd and

$$\lim_{|u|\to\infty} \frac{f(u)}{u} = b, \quad 0 < \lambda_1 < b;$$

(f_2) $f(s) = o(s)$ as $s \to 0$;

(f_3) $\dfrac{f(s)}{s} < f'(s)$, if $s > 0$;

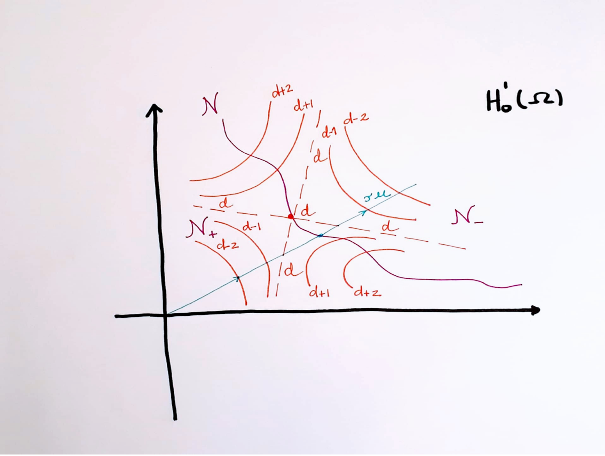
(f_4) $f(s)s - 2F(s) > 0$, if $s \neq 0$ and $\lim_{|s|\to\infty} f(s)s - 2F(s) = +\infty$, where $F(s) = \int_0^s f(\xi)d\xi$.

Some parabolic background: asymptoticaly linear

Level curves of the functional

Some parabolic background: asymptoticaly linear

Level curves of the functional



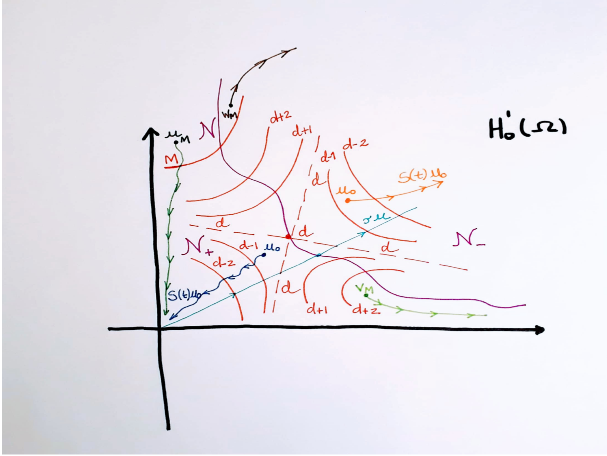
Nehari manifold $\mathcal{N}$ with complementary sets $\mathcal{N}_-$ and $\mathcal{N}_+$. d is the ground state energy.

Some parabolic background: asymptoticaly linear

Evolutionary dynamics

Some parabolic background: asymptoticaly linear

Evolutionary dynamics



F. and Maia 21

Degenerate logistic type parabolic equation

$$\partial_t u = \Delta u + \lambda u + b(x) f(u), \quad (x,t) \text{ in } \Omega \times \mathbb{R}^+$$

Assume $f \in C^1(\mathbb{R})$ is odd

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Assume $f \in C^1(\mathbb{R})$ is odd

(f_1) $\lim_{s \to 0} \frac{f(s)}{s} = 0$;

(f_2) Let $F(s) = \int_0^s f(t)dt$, $\lim_{s \to +\infty} \frac{F(s)}{s^2} = +\infty$;

(f_3) $\dfrac{f(s)}{s} < f'(s)$, if $s > 0$;

(f_4) There exists $2 < q < 2^*$, $a_1 < 0$ and an integer $k = \{0, 1\}$ such that $|f^{(k)}(s)| \leq a_1(1 + |s|^{q-(k+1)})$.

Stationary problem: existence and nonexistence

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Consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ of class C^2 given by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda \int_\Omega u^2 - \int_\Omega b(x)F(u)dx. \tag{1}$$

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Critical points u of the functional are weak solutions of the stationary problem:

$$I'(u)v = \int_\Omega (\nabla u \cdot \nabla v - \lambda uv)dx - \int_\Omega b(x)f(u)vdx = 0 \ ,$$

for $u, v \in H_0^1(\Omega)$.

Stationary problem: existence and nonexistence

Nehari manifold

The so called Nehari manifold

$$\mathcal{N} := \{u \in H_0^1(\Omega) : u \neq 0, \quad J(u) = 0\},$$

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Also take the complementary sets

$$\mathcal{N}_+ := \{u \in H_0^1(\Omega) : u \neq 0, \quad J(u) > 0\}$$

and

$$\mathcal{N}_- := \{u \in H_0^1(\Omega) : u \neq 0, \quad J(u) < 0\}.$$

Stationary problem: existence and nonexistence

Theorem

(i) *If* $\Omega_0 = \emptyset$*, then for every* $\lambda > 0$ *there exists a unique positive stationary solution.*

(ii) *If* $\Omega_0 \neq \emptyset$*, then for every* $\lambda < \lambda_1(\Omega_0)$ *there exists a unique positive stationary solution.*

(iii) *If* $\Omega_0 \neq \emptyset$ *and* $\lambda \geq \lambda_1(\Omega_0)$*, the problem admits no positive solution.*

- Ouyang 91
- Alama and Tarantello 93
- Cardoso, Furtado and Maia 24

The points in the Nehari manifold $\mathcal{N}$ correspond to stationary points of the maps

$$\phi_u : t \mapsto J(tu)$$

and so it is natural to divide $\mathcal{N}$ into three subsets corresponding to local minima, local maxima and points of inflexion: $\mathcal{S}^+, \mathcal{S}^-$ and $\mathcal{S}^0$.

Stationary problem: existence and nonexistence

Fibrering maps

Theorem (F. and Maia 24)

Suppose $f(u) = u^{p-1}u$, *with* $1 < p < 2^* - 1$, $2^* = +\infty$, *if* $N = 2$, *and* $2^* = 2N/N-2$, *if* $N \geq 3$. *Then, for* $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$.

(i) $\mathcal{S}^0 = \{0\}$;

(ii) $\mathcal{S}^- = \emptyset$; *and*

(iii) $\mathcal{S}^+$ *is bounded,*

Stationary problem: existence and nonexistence

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Remark: If $u \in \mathcal{N}$, it holds that

- $u \in \mathcal{S}^+$ if and only if $I(u) < 0$;
- $u \in \mathcal{S}^-$ if and only if $I(u) > 0$;
- $u \in \mathcal{S}_0$ if and only if $I(u) = 0$.

Stationary problem: existence and nonexistence

Ground state solution

Theorem (F. and Maia 24)

If $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$, then I is bounded from below in $H_0^1(\Omega)$, and there exists a minimizer $\varphi > 0$ such that $I(\varphi) = \underline{d}$. Additionally, φ is isolated from other stationary solutions with respect to the $H_0^1(\Omega)$ topology.

Lemma

If $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ and $u \in \mathcal{N}_-$ then u is projectable on $\mathcal{S}^+$, i.e., there exists t_u such that $t_u u \in \mathcal{S}^+$. Moreover,

- $I(u) < 0$
- *the set $\mathcal{N}_-$ is bounded in $H_0^1(\Omega)$.*

Stationary problem: existence and nonexistence

Mountain Pass Theorem on $\mathcal{N}$

Theorem (F. and Maia 24)

Let $\tilde{I}(u) := I(u) - \underline{d}$. For $\varphi > 0$ and $-\varphi < 0$ local minima on $\mathcal{S}^+$, then $\tilde{I}$ satisfies the geometrical hypotheses of the Mountain Pass Theorem on $\mathcal{N}$.

Moreover $\tilde{I}$ satisfies $(PS)_c$ condition at

$$d = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma : [0,1] \to \mathcal{N} \ : \gamma(0) = \varphi, \gamma(1) = -\varphi\}$, and so there exists a solution u^* satisfying $I(u^*) = d > \underline{d}$.

Stationary problem: existence and nonexistence

Mountain Pass Theorem on $\mathcal{N}$

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Note that the solution just found may be trivial. The next theorem gives a sufficient condition in order to obtain a **nontrivial** min-max solution u^*.

Stationary problem: existence and nonexistence

Sign changing solution

Theorem (F. and Maia 24)

Let $\lambda_1(\Omega) < \lambda_2(\Omega) < \lambda < \lambda_1(\Omega_0)$. Then there exists a mountain pass solution u^* of the stationary problem , which is sign-changing with energy level $\underline{d} < I(u^*) = d < 0$. Moreover, u^* has Morse index at least 1, and, therefore, it is an unstable solution.

Cardoso, Furtado and Maia 24: general superlinear nonlinearity

Sketch of the proof

- Want to show that $I(u^*) = d < 0$, which gives $u^* \neq 0$.
- Consider the positive (normalized in $L^2(\Omega)$) eigenfunctions of $-\Delta$:
  - Ω: ϕ_1, ϕ_2
  - Ω_0: ϕ_1^0, ϕ_2^0
- In order to construct a convenient path in Γ not passing through zero we define

$$w = t_1(\phi_1 + \varepsilon\phi_1^0) + t_2(\phi_2 + \varepsilon\phi_2^0)$$

with constants $t_1, t_2 > 0$, and $\varepsilon > 0$ to be chosen sufficiently small. Using that $b(x) \leq 0$, there is a positive constant C such that

Sketch of the proof

$$I(w) \leq \frac{(t_1^2 + t_2^2)}{2} \max\{\lambda_1(\Omega) - \lambda, \lambda_2(\Omega) - \lambda\} + O(\varepsilon(t_1^2 + t_2^2)) \leq -\delta_1 < 0,$$

for some constant $\delta_1 > 0$.
Note that

- $\|w\| \leq \sqrt{t_1^2 + t_2^2 + C\varepsilon^2}$, taking $t_1, t_2 > 0$ and ε sufficiently small
- $\nu + 1 > 2$, and
- $\lambda_1(\Omega) < \lambda_2(\Omega) < \lambda$.

Sketch of the proof

Define $w_1 := t_1(\phi_1 + \varepsilon\phi_1^0)$ and $w_2 := t_2(\phi_2 + \varepsilon\phi_2^0)$, and $w_\theta := \cos(\theta)w_1 + \sin(\theta)w_2$,

$$I(w_\theta) < 0; \quad \forall\theta \in [0, \pi]$$

Define the path:

$$\gamma(s) := \begin{cases} [(1-3s)\varphi + 3s(w_1)], & s \in [0, 1/3] \\ w_{\theta(s)}, & s \in [1/3, 2/3] \text{ and } \theta(s) = 3(s - 1/3)\pi, \\ [3(1-s)(-w_1) + 3(s-2/3)(-\varphi)], & s \in [2/3, 1], \end{cases} \tag{2}$$

which can be projected on $\mathcal{N}$ by the multiplication $\tau(s)\gamma(s)$, with

$$\tau(s) = \left[\frac{\int_\Omega |\nabla\gamma(s)|^2 - \lambda(\gamma(s))^2}{\int_\Omega b(x)|\gamma(s)|^{\nu+1}}\right].$$

Sketch of the proof

- There is a negative upper bound in $\mathcal{S}^+$

$$I(\tau(s)\gamma(s)) \leq \max_{0\leq t\leq 1} I(\tau(s)\gamma(s)) < 0$$

  for all $s \in [0, 1]$.

- By the definition of the min-max level d, it follows that $I(u^*) = d < 0$.
- Hopf Lemma and uniqueness of positive solutions assure u^* is a sign changing solution.

Main parabolic results: $f(u) = |u|^{p-1}u$

- Global existence in time: the equation has a corresponding semigroup $S(t)$ acting in L^2 and defined by

$$S(t)u_0(\cdot) = u(\cdot, t), \quad t \geq 0.$$

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- Global existence in time: the equation has a corresponding semigroup $S(t)$ acting in L^2 and defined by

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- In particular, the set of (finite-time) blow-up solutions is empty.
- If we differentiate the map $t \mapsto I(u(t))$ with respect to t, we get

$$\frac{d}{dt} I(u(t)) = -\int_\Omega u_t^2(t) \quad \text{for all } t > 0, \tag{3}$$

which implies that I is decreasing along non-stationary solutions (Lyapunov gradient structure).

Main parabolic results: $f(u) = |u|^{p-1}u$

No blow-up and no grow-up

Theorem (F. and Maia 22)

Let $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. Then the solutions $u(x,t)$ exist for all forward time. Additionally, no solution may blow-up in infinite-time, i.e., no grow-up.

Main parabolic results: $f(u) = |u|^{p-1}u$

Positive solutions

Given a nonnegative u_0, then

$$u(t, u_0) \to \varphi,$$

as $t \to \infty$, for φ the unique positive stationary solution.

Remark: The trivial solution is an isolated equilibrium point and is known to be unstable in the subset of nonnegative initial data, if $\lambda_1(\Omega) < \lambda$.

Main parabolic results: $f(u) = |u|^{p-1}u$

Stability of Mountain Pass solution

Theorem (F. and Maia 22)

There exist initial data $u_0, v_0 \in H_0^1(\Omega)$ with $u_0 \in \mathcal{N}_+$ and $v_0 \in \mathcal{N}_-$, satisfying

$$u(t, u_0) \to u^*, \ u(t, v_0) \to u^*,$$

as $t \to \infty$ $(u_0, v_0 \in W^s_{loc}(u^*))$.

Main parabolic results: $f(u) = |u|^{p-1}u$

Stability of Mountain Pass solution

- The linearized operator at u^*

$$\mathcal{L}u = -\Delta u - f'(u^*)u \tag{4}$$

is self-adjoint in $L^2(\Omega)$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$ and spectrum entirely composed of eigenvalues.

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- $\{\mu_i\}_{i=1}^{\infty}$ the nondecreasing sequence of eigenvalues of $\mathcal{L}$, repeated according to their (finite) multiplicities, and $\{\psi_i\}_{i=1}^{\infty}$ the eigenfunctions.

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- $\{\mu_i\}_{i=1}^{\infty}$ the nondecreasing sequence of eigenvalues of $\mathcal{L}$, repeated according to their (finite) multiplicities, and $\{\psi_i\}_{i=1}^{\infty}$ the eigenfunctions.
- We have $\mu_i \to +\infty$ as $i \to \infty$ and $\mu_1 < 0$.

Lemma

For u^* there exists $i > q$, such that

$$a_i := \int_\Omega \phi\psi_i \neq 0. \tag{5}$$

where $q := \max\{i \in \mathbb{N} : \mu_i \leq 0\}$.

We may decompose any u^* by

$$u^* = \sum_{i=1}^{q} a_i\psi_i + \sum_{i=q+1}^{\infty} a_i\psi_i.$$

Main parabolic results: $f(u) = |u|^{p-1}u$

Stability of Mountain Pass solution

- X_1 the finite dimensional subspace of $H_0^1(\Omega)$ spanned by $\{\psi_i : 1 \leq i \leq q\}$ and X_2 spanned by $\{\psi_i : i > q\}$

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Stability of Mountain Pass solution

- X_1 the finite dimensional subspace of $H_0^1(\Omega)$ spanned by $\{\psi_i : 1 \leq i \leq q\}$ and X_2 spanned by $\{\psi_i : i > q\}$
- The local stable manifold of u^* is tangent to X_2 at u^*. In addition, there exists a neighborhood V of 0 in X_2 and a C^1 map $h : V \to X_1$ such that
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- By Taylor's expansion at u^*, since $J(u^*) = 0$ then $J(u_0) = J(u^*) + J'(u^*)(\epsilon\psi_i) + R_\epsilon = \epsilon\mu_i a_i + R_\epsilon \quad (a_i > 0)$.

Summarizing

- Apply variational methods and comparison principle in order to analyze the behavior of solutions.
- The Nehari manifold is used to locate the stationary solutions, and identify convergence regions of evolutionary trajectories.
- To our knowledge, this is the first time the Nehari approach is applied to address the asymptotic analysis of solutions to the logistic problem.
- Exploit further the geometric features of the associated Nehari manifold. Since stationary solutions play a crucial role in the description of the evolution, several elliptic tools turn out to be quite useful for our purposes.

Thank you!