Boundary Weak Harnack Estimates and Regularity for Elliptic Operators in Divergence Form and Applications in PDEs

5^{th} Mostly Maximum Principle
June 24-28, 2024

Mayra Soares

Department of Mathematics
University of Brasilia

Joint work with Boyan Sirakov and Fiorella Rendón - PUC-Rio

Introduction

We consider general uniformly elliptic equations in **divergence form**, under the **weakest assumptions** on the leading coefficients and on the boundary of the domain.

We obtain a **global extension** of the classical **Weak Harnack Inequality,** which extends and quantifies the **Hopf Boundary Point Lemma**.

Our main tool are the **global $\mathcal{C}^1$-estimates** and suitable **barrier functions**, which are solutions of auxiliaries problems.

We provide an application showing how to use these results to deduce **a priori bounds and multiplicity of solutions** for a class of quasilinear elliptic problems.

Motivation

- We consider **nonnegative weak supersolutions** of the problem

$$\mathcal{L}u = f(x),\ x \in \Omega, \tag{$\mathcal{P}$}$$

$$\mathcal{L}u := -\mathrm{div}(A(x)Du + \beta u) + b(x) \cdot Du + c(x)u,\ x \in \Omega, \tag{$\mathcal{L}$}$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded domain, under certain regularity assumptions.

Motivation

- We consider **nonnegative weak supersolutions** of the problem

$$\mathcal{L}u = f(x),\ x \in \Omega, \qquad (\mathcal{P})$$

$$\mathcal{L}u := -\text{div}(A(x)Du + \beta u) + b(x) \cdot Du + c(x)u,\ x \in \Omega, \qquad (\mathcal{L})$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded domain, under certain regularity assumptions.

- The De Giorgi-Moser **"Weak Harnack Inequality" (WHI)** is an interior result stated for any nonnegative supersolution of $(\mathcal{P})$ as

$$\left(\int_{B_R} u^\varepsilon \right)^{\frac{1}{\varepsilon}} \leq C_0 \left(\inf_{B_R} u + ||f||_{\mathcal{L}^p(B_{2R})} \right) \text{ for } \varepsilon < \frac{n}{(n-2)^+}, \qquad \text{(WHI)}$$

where $B_{2R} = B_{2R}(x_0) \subset \Omega$ and $C_0 = C_0(n, \vartheta, p, q, R, \varepsilon, \beta, b, c)$.

Motivation

- In [1] Sirakov proved a **global extension** to the (WHI) in terms of the **distance up to the boundary** $d = d(x, \partial\Omega)$, for **non-divergence** form operators.

✎ **[1] Sirakov, B. Boundary Harnack Estimates and Quantitative Strong Maximum Principles for Uniformly Elliptic PDE, *Int. Math. Res. Notices*, no 24, 7457-7482, 2018.**

Motivation

- In [1] Sirakov proved a **global extension** to the (WHI) in terms of the **distance up to the boundary** $d = d(x, \partial\Omega)$, for **non-divergence** form operators.

  ✎ **[1] Sirakov, B. Boundary Harnack Estimates and Quantitative Strong Maximum Principles for Uniformly Elliptic PDE, *Int. Math. Res. Notices*, no 24, 7457-7482, 2018.**

- It was called **Boundary Weak Harnack Inequality - (bWHI).**

Motivation

- In [1] Sirakov proved a **global extension** to the (WHI) in terms of the **distance up to the boundary** $d = d(x, \partial\Omega)$, for **non-divergence** form operators.

  ✎ **[1] Sirakov, B. Boundary Harnack Estimates and Quantitative Strong Maximum Principles for Uniformly Elliptic PDE, *Int. Math. Res. Notices*, no 24, 7457-7482, 2018.**

- It was called **Boundary Weak Harnack Inequality - (bWHI).**

- Inspired by [1], we obtained a version of the (bWHI) for **divergence-type** equations considering **optimal regularity assumptions.**

Regularity Assumptions

- We say that $\sigma : [0,1] \to \mathbb{R}_+$ is a **Dini function** and write $\sigma \in \mathcal{D}$ if

(i) $\sigma(0) = 0 < \sigma(t)/2 \leq \sigma(s) \leq \sigma(t)$ for $0 < t/2 \leq s \leq t$;
(ii) $\sigma(\tau)/\tau$ is non-increasing and $\int_0^s \frac{\sigma(\tau)}{\tau} d\tau < +\infty$.

Regularity Assumptions

- We say that $\sigma : [0,1] \to \mathbb{R}_+$ is a **Dini function** and write $\sigma \in \mathcal{D}$ if

  (i) $\sigma(0) = 0 < \sigma(t)/2 \leq \sigma(s) \leq \sigma(t)$ for $0 < t/2 \leq s \leq t$;
  (ii) $\sigma(\tau)/\tau$ is non-increasing and $\int_0^s \frac{\sigma(\tau)}{\tau} d\tau < +\infty$.

- We say $\psi : \Omega \to \mathbb{R}$ is **Dini continuous** function in Ω and write $\psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$ if there exists some $\sigma \in \mathcal{D}$ such that

$$|\psi(x) - \psi(y)| \leq \sigma(|x-y|) \quad \text{for all} \quad x, y \in \overline{\Omega}.$$

Regularity Assumptions

- We say that $\sigma : [0,1] \to \mathbb{R}_+$ is a **Dini function** and write $\sigma \in \mathcal{D}$ if

  (i) $\sigma(0) = 0 < \sigma(t)/2 \leq \sigma(s) \leq \sigma(t)$ for $0 < t/2 \leq s \leq t$;
  (ii) $\sigma(\tau)/\tau$ is non-increasing and $\int_0^s \frac{\sigma(\tau)}{\tau} d\tau < +\infty$.

- We say $\psi : \Omega \to \mathbb{R}$ is **Dini continuous** function in Ω and write $\psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$ if there exists some $\sigma \in \mathcal{D}$ such that

$$|\psi(x) - \psi(y)| \leq \sigma(|x-y|) \quad \text{for all} \quad x, y \in \overline{\Omega}.$$

- We say Ω is a $\mathcal{C}^{1,\mathcal{D}ini}$ **domain** if, locally, $\partial\Omega$ **can be seen as the graph of a** $\mathcal{C}^1$**-function**, whose derivatives are of class $\mathcal{C}^{0,\mathcal{D}ini}$.

Regularity Assumptions

- We say that $\sigma : [0,1] \to \mathbb{R}_+$ is a **Dini function** and write $\sigma \in \mathcal{D}$ if

  (i) $\sigma(0) = 0 < \sigma(t)/2 \le \sigma(s) \le \sigma(t)$ for $0 < t/2 \le s \le t$;
  (ii) $\sigma(\tau)/\tau$ is non-increasing and $\int_0^s \frac{\sigma(\tau)}{\tau} d\tau < +\infty$.

- We say $\psi : \Omega \to \mathbb{R}$ is **Dini continuous** function in Ω and write $\psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$ if there exists some $\sigma \in \mathcal{D}$ such that

$$|\psi(x) - \psi(y)| \le \sigma(|x-y|) \quad \text{for all} \quad x, y \in \overline{\Omega}.$$

- We say Ω is a $\mathcal{C}^{1,\mathcal{D}ini}$ **domain** if, locally, $\partial\Omega$ **can be seen as the graph of a** $\mathcal{C}^1$**-function**, whose derivatives are of class $\mathcal{C}^{0,\mathcal{D}ini}$.

- Setting $B_R^+ = B_R \cap \Omega$, we say a function ψ has **Dini mean oscillation on** Ω and write $\psi \in \mathcal{C}^{0,m\mathcal{D}ini}(\Omega)$ if there exists $\sigma_m \in \mathcal{D}$ such that

$$\fint_{B_R^+(x)} |\psi(y) - \fint_{B_R^+(x)} \psi(z)dz|dy \le \sigma_m(R) \quad \text{for every} \quad R > 0, \; x \in \overline{\Omega}.$$

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.
- Choose $\sigma(\tau) = C|\tau|^{\alpha}$ for a suitable constant $C > 0$.

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.
- Choose $\sigma(\tau) = C|\tau|^{\alpha}$ for a suitable constant $C > 0$.

Then, Hölder continuity $\implies$ Dini continuity.

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.
- Choose $\sigma(\tau) = C|\tau|^\alpha$ for a suitable constant $C > 0$.

Then, Hölder continuity $\implies$ Dini continuity.

- Note that for every $R > 0$, $x \in \overline{\Omega}$, it follows that

$$\fint_{B_R^+(x)} |\psi(y) - \fint_{B_R^+(x)} \psi(z)dz|dy \leq \sup_{y,z \in B_R^+(x)} |\psi(y) - \psi(z)|.$$

Then, Dini continuity $\implies$ Dini mean oscillation.

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.
- Choose $\sigma(\tau) = C|\tau|^{\alpha}$ for a suitable constant $C > 0$.

Then, Hölder continuity $\implies$ Dini continuity.

- Note that for every $R > 0,\ x \in \overline{\Omega}$, it follows that

$$\fint_{B_R^+(x)} |\psi(y) - \fint_{B_R^+(x)} \psi(z)dz|dy \leq \sup_{y,z\in B_R^+(x)} |\psi(y) - \psi(z)|.$$

Then, Dini continuity $\implies$ Dini mean oscillation.

- A standard example of **non-Dini continuous** function which has Dini mean oscillation is $\psi(x) = |\log|x||^{-\gamma}, \gamma \in (0,1]$.

A Weaker Condition

- For all $\alpha \in (0,1)$, $\psi \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \implies \psi \in \mathcal{C}^{0,\mathcal{D}ini}(\Omega)$.
- Choose $\sigma(\tau) = C|\tau|^\alpha$ for a suitable constant $C > 0$.

Then, Hölder continuity $\implies$ Dini continuity.

- Note that for every $R > 0,\ x \in \overline{\Omega}$, it follows that

$$\fint_{B_R^+(x)} |\psi(y) - \fint_{B_R^+(x)} \psi(z)dz|dy \leq \sup_{y,z \in B_R^+(x)} |\psi(y) - \psi(z)|.$$

Then, Dini continuity $\implies$ Dini mean oscillation.

- A standard example of **non-Dini continuous** function which has Dini mean oscillation is $\psi(x) = |\log|x||^{-\gamma}, \gamma \in (0,1]$.

Then, Dini mean oscillation $\not\Longrightarrow$ Dini continuity.

Our Setting

- We consider **nonnegative weak supersolutions** of the problem

$$\mathcal{L}u = f(x),\ x \in \Omega, \qquad (\mathcal{P})$$

$$\mathcal{L}u := -\mathrm{div}(A(x)Du + \beta u) + b(x) \cdot Du + c(x)u,\ x \in \Omega, \qquad (\mathcal{L})$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded $\mathcal{C}^{1,\mathcal{D}ini}$ domain.

Our Setting

- We consider **nonnegative weak supersolutions** of the problem

$$\mathcal{L}u = f(x),\ x \in \Omega, \qquad (\mathcal{P})$$

$$\mathcal{L}u := -\text{div}(A(x)Du + \beta u) + b(x)\cdot Du + c(x)u,\ x \in \Omega, \qquad (\mathcal{L})$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded $\mathcal{C}^{1,\mathcal{D}ini}$ domain.

- Operator $\mathcal{L}$ is uniformly elliptic and $A(x) = (a_{i,j}(x))$ is a symmetric matrix, satisfying $a_{i,j} \in \mathcal{C}^{0,m\mathcal{D}ini}(\Omega)$, i. e. having a **Dini mean oscillation** in $\Omega_{d_0} = \{x \in \Omega : d(x,\partial\Omega) < d_0\}$, for all $i,j = 1,...,n$ and

$$\vartheta I_n \leq A(x) \leq \vartheta^{-1} I_n \text{ in } \Omega, \qquad (0.1)$$

for some $\vartheta > 0, d_0 > 0$, where $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator.

Our Setting

- We consider **nonnegative weak supersolutions** of the problem
$$\mathcal{L}u = f(x),\ x \in \Omega, \qquad (\mathcal{P})$$
$$\mathcal{L}u := -\mathrm{div}(A(x)Du + \beta u) + b(x) \cdot Du + c(x)u,\ x \in \Omega, \qquad (\mathcal{L})$$
where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded $\mathcal{C}^{1,\mathcal{D}ini}$ domain.

- Operator $\mathcal{L}$ is uniformly elliptic and $A(x) = (a_{i,j}(x))$ is a symmetric matrix, satisfying $a_{i,j} \in \mathcal{C}^{0,m\mathcal{D}ini}(\Omega)$, i. e. having a **Dini mean oscillation** in $\Omega_{d_0} = \{x \in \Omega : d(x, \partial\Omega) < d_0\}$, for all $i, j = 1, ..., n$ and
$$\vartheta I_n \leq A(x) \leq \vartheta^{-1} I_n \text{ in } \Omega, \qquad (0.1)$$
for some $\vartheta > 0, d_0 > 0$, where $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator.

- For some $q > n$ and some $p > n/2$, we also require that
$$\beta, |b| \in \mathcal{L}^q(\Omega),\ c \geq 0 \text{ in } \Omega,\ \text{and } c, f \in \mathcal{L}^p(\Omega). \qquad (0.2)$$

Main Result

Under our setting, we obtain the following

Boundary Weak Harnack Inequality - (bWHI)

$$\left(\fint_{B_R^+}\left(\frac{u}{d}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}} \leq C\left(\inf_{B^+}\frac{u}{d} + ||f||_{\mathcal{L}^p(B_{2R}^+)}\right) \text{ for some } \varepsilon > 0, \quad \text{(bWHI)}$$

for **nonnegative weak supersolutions** u of problem $(\mathcal{P})$ and for every $x_0 \in \partial\Omega, R \leq d_0/2$ and $B_R^+ = B_R(x_0) \cap \Omega$.

A Previous Result

- In [2], Sirakov developed global estimates for the following uniformly elliptic PDEs in divergence-form,

$$-\text{div}(A(x)Du) + b(x) \cdot Du \geq -f$$

in a $\mathcal{C}^{1,1}$**-domain** Ω **and for a matrix** $A(x) \in \mathcal{W}^{1,q}(\Omega)$ and functions $b, f \in \mathcal{L}^q(\Omega)$, para $q > n$.

✎ **[2] Sirakov, B. Global integrability and weak Harnack estimates for elliptic PDE in divergence form, Anal. PDE 15 (1) 2849–2868, 2022.**

A Previous Result

- In [2], Sirakov developed global estimates for the following uniformly elliptic PDEs in divergence-form,

$$-\text{div}(A(x)Du) + b(x) \cdot Du \geq -f$$

in a $\mathcal{C}^{1,1}$**-domain** Ω **and for a matrix** $A(x) \in \mathcal{W}^{1,q}(\Omega)$ and functions $b, f \in \mathcal{L}^q(\Omega)$, para $q > n$.

✎ **[2] Sirakov, B. Global integrability and weak Harnack estimates for elliptic PDE in divergence form, Anal. PDE 15 (1) 2849–2868, 2022.**

- His main result was the Boundary Weak Harnack Inequality,

$$\inf_{\Omega} \frac{u}{d} \geq C \left(\int_{\Omega} \left(\frac{u}{d} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}} - C||f||_{\mathcal{L}^q(\Omega)},$$

and the guarantee of the **best constant of integrability:** $\varepsilon < 1$**.**

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.

- This is not our case, due to the low regularity.

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.
- This is not our case, due to the low regularity.
- Sirakov adopted a clever approach that allowed to produce a **Moser-type iterative argument.**

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.

- This is not our case, due to the low regularity.

- Sirakov adopted a clever approach that allowed to produce a **Moser-type iterative argument.**

- The regularity of the coefficients $A(x) \in \mathcal{W}^{1,q}(\Omega)$ **was essential** for applying the Divergence Theorem and make his arguments work.

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.
- This is not our case, due to the low regularity.
- Sirakov adopted a clever approach that allowed to produce a **Moser-type iterative argument.**
- The regularity of the coefficients $A(x) \in \mathcal{W}^{1,q}(\Omega)$ **was essential** for applying the Divergence Theorem and make his arguments work.
- Unluckily, this method **cannot be adapted** under our assumptions.

Optimal Integrability

- The optimal exponent $\varepsilon < 1$, provided in [2], was specified for operators which can be **written in divergence and non-divergence form**.
- This is not our case, due to the low regularity.
- Sirakov adopted a clever approach that allowed to produce a **Moser-type iterative argument.**
- The regularity of the coefficients $A(x) \in \mathcal{W}^{1,q}(\Omega)$ **was essential** for applying the Divergence Theorem and make his arguments work.
- Unluckily, this method **cannot be adapted** under our assumptions.
- Therefore, it is **still an OPEN QUESTION** the optimal exponent $\varepsilon > 0$ for the global integrability of u **under our sharp hypotheses.**

Conclusions Extracted from the (bWHI)

- The (bWHI) **quantifies the positivity of the supersolution u close to $\partial\Omega$**, as well as, the (WHI) quantifies the positivity of u in the interior.

Conclusions Extracted from the (bWHI)

- The (bWHI) **quantifies the positivity of the supersolution u close to $\partial\Omega$**, as well as, the (WHI) quantifies the positivity of u in the interior.

- The (bWHI) also quantifies the Boundary Point Principle (BPP).

Conclusions Extracted from the (bWHI)

- The (bWHI) **quantifies the positivity of the supersolution u close to $\partial\Omega$**, as well as, the (WHI) quantifies the positivity of u in the interior.
- The (bWHI) also quantifies the Boundary Point Principle (BPP).
- For the **homogeneous equation**, the Boundary Weak Harnack Inequality gives

$$\frac{u}{d} \geq \inf_{B_R^+} \frac{u}{d} \geq C \left(\int_{B_R^+} \left(\frac{u}{d}\right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}, \text{ in } B_R^+ = B_R^+(x_0),$$

which, passing to the limit with $x \to x_0$, implies the (BPP).

Conclusions Extracted from the (bWHI)

- The (bWHI) **quantifies the positivity of the supersolution u close to $\partial\Omega$**, as well as, the (WHI) quantifies the positivity of u in the interior.

- The (bWHI) also quantifies the Boundary Point Principle (BPP).

- For the **homogeneous equation**, the Boundary Weak Harnack Inequality gives
$$\frac{u}{d} \geq \inf_{B_R^+} \frac{u}{d} \geq C\left(\int_{B_R^+} \left(\frac{u}{d}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}}, \text{ in } B_R^+ = B_R^+(x_0),$$
which, passing to the limit with $x \to x_0$, implies the (BPP).

- If $\mathcal{L}u \geq 0$ in B_{2R}^+ and $u \geq c_0 d$ in some $\omega \subset B_R^+$ with $|\omega| > 0$, the (bWHI) implies that
$$u \geq \kappa c_0 d, \quad \textbf{in the whole } B_R^+,$$
for some $\kappa > 0$ depending only on $|\omega|$ and the data, **quantifying the positivity of u close to $\partial\Omega$**.

Quantifying the Boundary Point Principle - (BPP)

- For the **nonhomogeneous case**, the quantification given by the (bWHI) preserves the integrability of u/d and **rectifies the (BPP) with the $\mathcal{L}^p$-norm of f**.

Quantifying the Boundary Point Principle - (BPP)

- For the **nonhomogeneous case**, the quantification given by the (bWHI) preserves the integrability of u/d and **rectifies the (BPP) with the $\mathcal{L}^p$-norm of f**.

- A lot of effort has been dedicated to getting **optimal conditions** for the validity of the (BPP).

Quantifying the Boundary Point Principle - (BPP)

- For the **nonhomogeneous case**, the quantification given by the (bWHI) preserves the integrability of u/d and **rectifies the (BPP) with the $\mathcal{L}^p$-norm of f**.

- A lot of effort has been dedicated to getting **optimal conditions** for the validity of the (BPP).

- The conditions vary in terms of the **regularity/geometry of the domain** and the **regularity/nature of the coefficients**.

Quantifying the Boundary Point Principle - (BPP)

- For the **nonhomogeneous case**, the quantification given by the (bWHI) preserves the integrability of u/d and **rectifies the (BPP) with the $\mathcal{L}^p$-norm of f**.

- A lot of effort has been dedicated to getting **optimal conditions** for the validity of the (BPP).

- The conditions vary in terms of the **regularity/geometry of the domain** and the **regularity/nature of the coefficients**.

- However, only recently the **importance of such a quantification** of the (BPP) has been recognized.

Quantifying the Boundary Point Principle - (BPP)

- For the **nonhomogeneous case**, the quantification given by the (bWHI) preserves the integrability of u/d and **rectifies the (BPP) with the $\mathcal{L}^p$-norm of f**.

- A lot of effort has been dedicated to getting **optimal conditions** for the validity of the (BPP).

- The conditions vary in terms of the **regularity/geometry of the domain** and the **regularity/nature of the coefficients**.

- However, only recently the **importance of such a quantification** of the (BPP) has been recognized.

- There are **no previous results quantifying the (BPP)** in such a way for divergence form equations, **not related to non-divergence ones.**

The General Statement of our (bWHI)

[3, Theorem 1.1] (Rendón-Sirakov-S.)

Assume $\mathcal{L}_i$ are uniformly elliptic operators under our assumptions, Ω is a $\mathcal{C}^{1,\mathcal{D}ini}$ domain and $f_i \in L^p(\Omega)$ for $p > \frac{n}{2}$, and $i = 1, 2$. Then,

✎ [3] Rendón, F., Sirakov, B. and S., M. Boundary weak Harnack estimates and regularity for elliptic PDE in divergence form. Nonlinear Anal. 235, Paper No. 113331, 2023.

The General Statement of our (bWHI)

[3, Theorem 1.1] (Rendón-Sirakov-S.)

Assume $\mathcal{L}_i$ are uniformly elliptic operators under our assumptions, Ω is a $\mathcal{C}^{1,\mathcal{D}ini}$ domain and $f_i \in \mathcal{L}^p(\Omega)$ for $p > \frac{n}{2}$, and $i = 1, 2$. Then,

(1) if $\mathcal{L}_1 u \geq f_1$ and $u \geq 0$ in Ω, there exist $\varepsilon, C > 0$, depending on the data, such that
$$\left(\int_\Omega \left(\frac{u}{d}\right)^\varepsilon\right)^{\frac{1}{\varepsilon}} \leq C\left(\inf_\Omega \frac{u}{d} + ||f_1||_{\mathcal{L}^p(\Omega)}\right).$$

✎ [3] Rendón, F., Sirakov, B. and S., M. Boundary weak Harnack estimates and regularity for elliptic PDE in divergence form. Nonlinear Anal. 235, Paper No. 113331, 2023.

The General Statement of our (bWHI)

[3, Theorem 1.1] (Rendón-Sirakov-S.)

Assume $\mathcal{L}_i$ are uniformly elliptic operators under our assumptions, Ω is a $\mathcal{C}^{1,\mathcal{D}ini}$ domain and $f_i \in \mathcal{L}^p(\Omega)$ for $p > \frac{n}{2}$, and $i = 1, 2$. Then,

(1) if $\mathcal{L}_1 u \geq f_1$ and $u \geq 0$ in Ω, there exist $\varepsilon, C > 0$, depending on the data, such that

$$\left(\int_\Omega \left(\frac{u}{d}\right)^\varepsilon\right)^{\frac{1}{\varepsilon}} \leq C\left(\inf_\Omega \frac{u}{d} + ||f_1||_{\mathcal{L}^p(\Omega)}\right).$$

(2) if $\mathcal{L}_1 u \geq f_1$, $\mathcal{L}_2 u \leq f_2$, $u \geq 0$ in Ω and $u \equiv 0$ on $\partial\Omega$, there exist $C > 0$, depending on the data, such that

$$\sup_\Omega \frac{u}{d} \leq C\left(\inf_\Omega \frac{u}{d} + ||f_1||_{\mathcal{L}^p(\Omega)} + ||f_2||_{\mathcal{L}^p(\Omega)}\right).$$

✎ [3] Rendón, F., Sirakov, B. and S., M. Boundary weak Harnack estimates and regularity for elliptic PDE in divergence form. Nonlinear Anal. 235, Paper No. 113331, 2023.

Our Method for Proving [3, Theorem 1.1]

- The statement of our (bWHI) is similar to that for the non-divergence case [1, Theorem 1.2].

Our Method for Proving [3, Theorem 1.1]

- The statement of our (bWHI) is similar to that for the non-divergence case [1, Theorem 1.2].
- However, the key point of our arguments **(the boundary growth lemma)** requires a **different approach**.

Our Method for Proving [3, Theorem 1.1]

- The statement of our (bWHI) is similar to that for the non-divergence case [1, Theorem 1.2].
- However, the key point of our arguments **(the boundary growth lemma)** requires a **different approach**.
- We use the classical idea of [4] to **compare u with a solution of a "frozen coefficients" equation** in a sufficiently small annulus, which touches the boundary.

✎ **[4] Finn, R. and Gilbarg, D. Asymptotic behavior and uniquenes of plane subsonic flows, Comm. Pure Appl. Math. 10 23–63, 1957.**

Our Method for Proving [3, Theorem 1.1]

- The statement of our (bWHI) is similar to that for the non-divergence case [1, Theorem 1.2].

- However, the key point of our arguments **(the boundary growth lemma)** requires a **different approach**.

- We use the classical idea of [4] to **compare u with a solution of a "frozen coefficients" equation** in a sufficiently small annulus, which touches the boundary.

  ✎ **[4] Finn, R. and Gilbarg, D. Asymptotic behavior and uniquenes of plane subsonic flows, Comm. Pure Appl. Math. 10 23–63, 1957.**

- We combine this comparison with the direct use of the **Maximum Principle** $(\checkmark)$ and the **global $\mathcal{C}^1$-estimates.**

$\mathcal{C}^1$-estimates up to the Boundary

[3, Theorem 3.1] (Dong–Escauriaza–Kim, [5])

Let Ω be a domain satisfying $\mathrm{diam}(\Omega) \leq 1$. If $u \in \mathcal{H}_0^1(\Omega)$ solves $\mathcal{L}u = f$ in Ω, under our assumptions, then $u \in \mathcal{C}^1(\overline{\Omega})$.

✎ **[5] Dong, H., L. Escauriaza, L. and Kim, S. On $\mathcal{C}^1$, $\mathcal{C}^2$ and weak type$-(1,1)$ estimates for linear elliptic operators: part II, Math. Ann. 370 447–489, 2018.**

$\mathcal{C}^1$-estimates up to the Boundary

[3, Theorem 3.1] (Dong–Escauriaza–Kim, [5])

Let Ω be a domain satisfying diam$(\Omega) \leq 1$. If $u \in \mathcal{H}_0^1(\Omega)$ solves $\mathcal{L}u = f$ in Ω, under our assumptions, then $u \in \mathcal{C}^1(\overline{\Omega})$.

In addition,
$$\|u\|_{\mathcal{C}^1(\overline{\Omega})} \leq C(\|u\|_{\mathcal{L}^2(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}),$$

where the constant $C > 0$ depends on the data.

✎ **[5] Dong, H., L. Escauriaza, L. and Kim, S. On $\mathcal{C}^1$, $\mathcal{C}^2$ and weak type$-(1,1)$ estimates for linear elliptic operators: part II, Math. Ann. 370 447–489, 2018.**

$\mathcal{C}^1$-estimates up to the Boundary

[3, Theorem 3.1] (Dong–Escauriaza–Kim, [5])

Let Ω be a domain satisfying $\mathrm{diam}(\Omega) \leq 1$. If $u \in \mathcal{H}_0^1(\Omega)$ solves $\mathcal{L}u = f$ in Ω, under our assumptions, then $u \in \mathcal{C}^1(\overline{\Omega})$.

In addition,
$$\|u\|_{\mathcal{C}^1(\overline{\Omega})} \leq C(\|u\|_{\mathcal{L}^2(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}),$$

where the constant $C > 0$ depends on the data.

Furthermore, there exists a modulus of continuity ω, depending on the Dini mean oscillation, such that $|Du(x) - Du(y)| \leq \omega(|x - y|)$.

✎ **[5] Dong, H., L. Escauriaza, L. and Kim, S. On $\mathcal{C}^1$, $\mathcal{C}^2$ and weak type$-(1,1)$ estimates for linear elliptic operators: part II, Math. Ann. 370 447–489, 2018.**

$\mathcal{C}^1$-estimates up to the Boundary

- It was crucial that the following **global $\mathcal{C}^1$-estimates** were available for the **standard Dirichlet problem** associated to our operator.

$\mathcal{C}^1$-estimates up to the Boundary

- It was crucial that the following **global $\mathcal{C}^1$-estimates** were available for the **standard Dirichlet problem** associated to our operator.

- Such a result, under the Dini mean oscillation condition, **has also just been stated by us in [3]**.

$\mathcal{C}^1$-estimates up to the Boundary

- It was crucial that the following **global $\mathcal{C}^1$-estimates** were available for the **standard Dirichlet problem** associated to our operator.

- Such a result, under the Dini mean oscillation condition, **has also just been stated by us in [3]**.

- It was inferred due to [5, Theorem 1.3, Lemma 2.11, Lemma 2.12] and by the standard **Sobolev bounds for weak solutions**,

$$\|Du\|_{\mathcal{L}^1(\Omega)} \leq \tilde{C}\|u\|_{\mathcal{H}^1(\Omega)} \leq C\big(\|u\|_{\mathcal{L}^2(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\big).$$

$\mathcal{C}^1$-estimates up to the Boundary

- It was crucial that the following **global $\mathcal{C}^1$-estimates** were available for the **standard Dirichlet problem** associated to our operator.
- Such a result, under the Dini mean oscillation condition, **has also just been stated by us in [3]**.
- It was inferred due to [5, Theorem 1.3, Lemma 2.11, Lemma 2.12] and by the standard **Sobolev bounds for weak solutions**,

$$\|Du\|_{\mathcal{L}^1(\Omega)} \leq \tilde{C}\|u\|_{\mathcal{H}^1(\Omega)} \leq C\big(\|u\|_{\mathcal{L}^2(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\big).$$

- The mere continuity on the leading coefficients **is NOT SUFFICIENT** to guarantee a $\mathcal{C}^1$-estimate up to the boundary.

More Contributions of the (bWHI)

- Our (bWHI) is **new even for $-\mathbf{div}(A(x)Du) \geq 0$ with Hölder continuous $\mathcal{C}^{0,\alpha}$** leading coefficients.

More Contributions of the (bWHI)

- Our (bWHI) is **new even for $-\textbf{div}(A(x)Du) \geq 0$ with Hölder continuous $\mathcal{C}^{0,\alpha}$** leading coefficients.
- In [6] the authors proved the **(BPP) for $-\textbf{div}(A(x)Du) \geq 0$, with Dini continuity of A.** It was the best available regularity, up to now.

✎ **[6] Apushkinskaya, D. E. and Nazarov, A. I. The normal derivative lemma and surrounding issues, Russian Math. Surveys 77 (2) 189–249, 2022.**

More Contributions of the (bWHI)

- Our (bWHI) is **new even for $-\textbf{div}(A(x)Du) \geq 0$ with Hölder continuous $\mathcal{C}^{0,\alpha}$** leading coefficients.

- In [6] the authors proved the **(BPP) for $-\textbf{div}(A(x)Du) \geq 0$, with Dini continuity of A.** It was the best available regularity, up to now.

  ✎ **[6] Apushkinskaya, D. E. and Nazarov, A. I. The normal derivative lemma and surrounding issues, Russian Math. Surveys 77 (2) 189–249, 2022.**

- Then, our (BPP) is also **new for non-Dini continuous** leading coefficients.

More Contributions of the (bWHI)

- Our (bWHI) is **new even for $-\mathbf{div}(A(x)Du) \geq 0$ with Hölder continuous $\mathcal{C}^{0,\alpha}$** leading coefficients.

- In [6] the authors proved the **(BPP) for $-\mathbf{div}(A(x)Du) \geq 0$, with Dini continuity of A.** It was the best available regularity, up to now.

  ✎ **[6] Apushkinskaya, D. E. and Nazarov, A. I. The normal derivative lemma and surrounding issues, Russian Math. Surveys 77 (2) 189–249, 2022.**

- Then, our (BPP) is also **new for non-Dini continuous** leading coefficients.

- Another consequence of the (bWHI) is the Boundary Regularity obtained under the assumptions of [3, Theorem 1.1 (2)].

Boundary Regularity Theory

[3, Theorem 1.2] (Rendón-Sirakov-S.)

Consider the elliptic operators $\mathcal{L}_1$ and $\mathcal{L}_2$, whose solutions of the Dirichlet problem in Ω have **uniformly continuous** gradient in $\overline{\Omega}$.

Boundary Regularity Theory

[3, Theorem 1.2] (Rendón-Sirakov-S.)

Consider the elliptic operators $\mathcal{L}_1$ and $\mathcal{L}_2$, whose solutions of the Dirichlet problem in Ω have **uniformly continuous** gradient in $\overline{\Omega}$.

If $u \in \mathcal{H}^1(\Omega)$ **is such that** $\mathcal{L}_1 u \geq f_1$**,** $\mathcal{L}_2 u \leq f_2$**,** for $f_1, f_2 \in \mathcal{L}^q(\Omega)$, then u **may not even be differentiable** in Ω, but it has a **uniformly continuous gradient on** $\partial\Omega$**.**

Boundary Regularity Theory

[3, Theorem 1.2] (Rendón-Sirakov-S.)

Consider the elliptic operators $\mathcal{L}_1$ and $\mathcal{L}_2$, whose solutions of the Dirichlet problem in Ω have **uniformly continuous** gradient in $\overline{\Omega}$.

If $u \in \mathcal{H}^1(\Omega)$ **is such that** $\mathcal{L}_1 u \geq f_1$**,** $\mathcal{L}_2 u \leq f_2$**,** for $f_1, f_2 \in \mathcal{L}^q(\Omega)$, then u **may not even be differentiable** in Ω, but it has a **uniformly continuous gradient on** $\partial\Omega$**.**
More precisely, under our setting with $\sigma = |\cdot|^\alpha$, there exist the "gradient" of u on $\partial\Omega$, $G \in \mathcal{C}^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$, and $C > 0$ such that

$$\|G\|_{\mathcal{C}^{0,\alpha}(\partial\Omega)} \leq C \left(\|u\|_{\mathcal{L}^\infty(\Omega)} + \|u\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)} + \sum \|f_i\|_{\mathcal{L}^q(\Omega)} \right) =: CW.$$

Boundary Regularity Theory

[3, Theorem 1.2] (Rendón-Sirakov-S.)

Consider the elliptic operators $\mathcal{L}_1$ and $\mathcal{L}_2$, whose solutions of the Dirichlet problem in Ω have **uniformly continuous** gradient in $\overline{\Omega}$.

If $u \in \mathcal{H}^1(\Omega)$ **is such that** $\mathcal{L}_1 u \geq f_1$, $\mathcal{L}_2 u \leq f_2$, for $f_1, f_2 \in \mathcal{L}^q(\Omega)$, then u **may not even be differentiable** in Ω, but it has a **uniformly continuous gradient on** $\partial\Omega$.

More precisely, under our setting with $\sigma = |\cdot|^\alpha$, there exist the "gradient" of u on $\partial\Omega$, $G \in \mathcal{C}^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$, and $C > 0$ such that

$$\|G\|_{\mathcal{C}^{0,\alpha}(\partial\Omega)} \leq C \left(\|u\|_{\mathcal{L}^\infty(\Omega)} + \|u\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)} + \sum \|f_i\|_{\mathcal{L}^q(\Omega)} \right) =: CW.$$

Furthermore, for each fixed $\hat{x}_0 \in \partial\Omega$, for every $x \in B^+_{1/2}(\hat{x}_0)$ and every $x_0 \in B_{1/2}(\hat{x}_0) \cap \partial\Omega$ we have

$$|u(x) - u(x_0) - G(x_0) \cdot (x - x_0)| \leq CW|x - x_0|^{1+\alpha}.$$

Boundary Regularity Theory

- [3, Theorem 1.2] is a consequence of our (bWHI), that guarantees the **boundary regularity** of a function u for the divergence framework.

Boundary Regularity Theory

- [3, Theorem 1.2] is a consequence of our (bWHI), that guarantees the **boundary regularity** of a function u for the divergence framework.

- For the non-divergence case, such a result is well known as **Krylov's property.**

Boundary Regularity Theory

- [3, Theorem 1.2] is a consequence of our (bWHI), that guarantees the **boundary regularity** of a function u for the divergence framework.
- For the non-divergence case, such a result is well known as **Krylov's property.**
- It is a consequence of his approach, introduced in [7], to prove the solvability and the regularity of the Dirichlet problem.

✎ **[7] Krylov, N.V. Boundedly inhomogeneous elliptic and parabolic equations in a domain, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1), 75–108, 1983.**

Boundary Regularity Theory

- [3, Theorem 1.2] is a consequence of our (bWHI), that guarantees the **boundary regularity** of a function u for the divergence framework.
- For the non-divergence case, such a result is well known as **Krylov's property.**
- It is a consequence of his approach, introduced in [7], to prove the solvability and the regularity of the Dirichlet problem.

  ✎ **[7] Krylov, N.V. Boundedly inhomogeneous elliptic and parabolic equations in a domain, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1), 75–108, 1983.**

- This is a fundamental result in the **non-divergence theory,** which has been extended and applied over the years by many authors.

Boundary Regularity Theory

- [3, Theorem 1.2] is a consequence of our (bWHI), that guarantees the **boundary regularity** of a function u for the divergence framework.
- For the non-divergence case, such a result is well known as **Krylov's property.**
- It is a consequence of his approach, introduced in [7], to prove the solvability and the regularity of the Dirichlet problem.

  ✎ **[7] Krylov, N.V. Boundedly inhomogeneous elliptic and parabolic equations in a domain, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1), 75–108, 1983.**

- This is a fundamental result in the **non-divergence theory,** which has been extended and applied over the years by many authors.
- However, this fact **has never been proven for pure divergence-form equations,** even in the simplest cases.

Applications

- The results in [1,2] have been instrumental in a **new method for producing a priori bounds** for positive solutions of nonlinear elliptic equations.

Applications

- The results in [1,2] have been instrumental in a **new method for producing a priori bounds** for positive solutions of nonlinear elliptic equations.
- Following this method, [3, Theorem 1.1] has been applied
  - In [8], to provide a priori bounds and **multiplicity of solutions of equations having quadratic dependence on the gradient.**

Applications

- The results in [1,2] have been instrumental in a **new method for producing a priori bounds** for positive solutions of nonlinear elliptic equations.

- Following this method, [3, Theorem 1.1] has been applied
  - In [8], to provide a priori bounds and **multiplicity of solutions of equations having quadratic dependence on the gradient.**
  - The generalization of many papers that apply the BPP, under stronger assumptions.

Applications

- The results in [1,2] have been instrumental in a **new method for producing a priori bounds** for positive solutions of nonlinear elliptic equations.

- Following this method, [3, Theorem 1.1] has been applied
  - In [8], to provide a priori bounds and **multiplicity of solutions of equations having quadratic dependence on the gradient.**
  - The generalization of many papers that apply the BPP, under stronger assumptions.

✎ **[8] F. Rendón and S., M. Multiplicity results for a class of quasilinear elliptic problems with quadratic growth on the gradient, preprint, arXiv:2207.10831.**

More About the Application in [8]

- We consider the following class of boundary value problems

$$\begin{cases} -\mathsf{div}(A(x)Du) & = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ \qquad u & \in \mathcal{H}_0^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \end{cases} \qquad (\mathcal{P}_\lambda)$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, is a $\mathcal{C}^{1,\mathcal{D}ini}$ **bounded domain.**

More About the Application in [8]

- We consider the following class of boundary value problems

$$\begin{cases} -\mathrm{div}(A(x)Du) & = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ u & \in \mathcal{H}_0^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \end{cases} \qquad (\mathcal{P}_\lambda)$$

  where $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, is a $\mathcal{C}^{1,\mathcal{D}ini}$ **bounded domain.**

- $c, h \in \mathcal{L}^p(\Omega)$ for some $p > n$, with functions $c^+,\ c^- \geq 0$ such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$.

More About the Application in [8]

- We consider the following class of boundary value problems

$$\begin{cases} -\mathrm{div}(A(x)Du) & = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ u & \in \mathcal{H}_0^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \end{cases} \qquad (\mathcal{P}_\lambda)$$

  where $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, is a $\mathcal{C}^{1,\mathcal{D}ini}$ **bounded domain.**

- $c, h \in \mathcal{L}^p(\Omega)$ for some $p > n$, with functions $c^+,\ c^- \geq 0$ such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$.

- $A(x) \in \mathcal{C}^{0,\mathcal{D}ini}$ is a uniformly positive bounded measurable matrix.

More About the Application in [8]

- We consider the following class of boundary value problems

$$\begin{cases} -\mathrm{div}(A(x)Du) & = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ u & \in \mathcal{H}_0^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \end{cases} \qquad (\mathcal{P}_\lambda)$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, is a $\mathcal{C}^{1,\mathcal{D}ini}$ **bounded domain.**

- $c, h \in \mathcal{L}^p(\Omega)$ for some $p > n$, with functions $c^+,\ c^- \geq 0$ such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$.

- $A(x) \in \mathcal{C}^{0,\mathcal{D}ini}$ is a uniformly positive bounded measurable matrix.

- $M(x)$ is a positive matrix such that

$$0 < \mu_1 I_n \leq M(x) \leq \mu_2 I_n \quad \text{in } \Omega, \qquad (0.3)$$

for some constants $\mu_1 > 0$ and $\mu_2 > 0$.

Coercive and Noncoercive Cases

- The class of problems $(\mathcal{P}_\lambda)$ is delicate to study, since **the gradient term has the same order as the Laplacian,** with respect to dilations.

Coercive and Noncoercive Cases

- The class of problems $(\mathcal{P}_\lambda)$ is delicate to study, since **the gradient term has the same order as the Laplacian,** with respect to dilations.

- The **coercive case, i.e.** $c \leq 0$, was introduced by Boccardo, Murat and Puel in the 80's, but the **uniqueness of solution** was proved in [10], many years latter.

  ✎ **[10] Arcoya, D., De Coster, C., Jeanjean, L. and Tanaka, K. Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl., 420, 772-780, 2014.**

Coercive and Noncoercive Cases

- The class of problems $(\mathcal{P}_\lambda)$ is delicate to study, since **the gradient term has the same order as the Laplacian,** with respect to dilations.

- The **coercive case, i.e.** $c \leq 0$, was introduced by Boccardo, Murat and Puel in the 80's, but the **uniqueness of solution** was proved in [10], many years latter.

  ✎ **[10] Arcoya, D., De Coster, C., Jeanjean, L. and Tanaka, K. Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl., 420, 772-780, 2014.**

- We consider the **noncoercive case,** $c \not\leq 0$, assuming that

$$\left\{ \begin{array}{rl} \Omega_{c^+} := \text{supp}(c^+), & |\Omega_{c^+}| > 0 \text{ and there exists} \\ \varepsilon > 0 \text{ such that } c^- = 0 & \text{in } \{x \in \Omega : d(x, \Omega_{c^+}) < \varepsilon\}, \end{array} \right. \qquad (\mathcal{A}_c^+)$$

and hence, the **uniqueness of solution is expected to fail.**

Our Strategy

- To prove our results, we have used our Boundary Weak Harnack Inequality to generalize the results in [12], under our setting.

  ✎ **[12] Sirakov, Boyan: A new method of proving a priori bounds for superlinear elliptic PDE, J. Math. Pures Appl. 141, 184-194, 2020.**

Our Strategy

- To prove our results, we have used our Boundary Weak Harnack Inequality to generalize the results in [12], under our setting.

  ✎ **[12] Sirakov, Boyan: A new method of proving a priori bounds for superlinear elliptic PDE, J. Math. Pures Appl. 141, 184-194, 2020.**

- We have showed that it is sufficient **to control the solutions on Ω_{c^+}.**

Our Strategy

- To prove our results, we have used our Boundary Weak Harnack Inequality to generalize the results in [12], under our setting.

  ✎ **[12] Sirakov, Boyan: A new method of proving a priori bounds for superlinear elliptic PDE, J. Math. Pures Appl. 141, 184-194, 2020.**

- We have showed that it is sufficient **to control the solutions on Ω_{c^+}.**

- Then, we have obtained **a uniform a priori upper bound** for the solutions of $(\mathcal{P}_\lambda)$ in a neighborhood of any fixed point $\overline{x} \in \overline{\Omega}_{c^+}$.

Our Strategy

- To prove our results, we have used our Boundary Weak Harnack Inequality to generalize the results in [12], under our setting.

  ✎ **[12] Sirakov, Boyan: A new method of proving a priori bounds for superlinear elliptic PDE, J. Math. Pures Appl. 141, 184-194, 2020.**

- We have showed that it is sufficient **to control the solutions on Ω_{c^+}.**

- Then, we have obtained **a uniform a priori upper bound** for the solutions of $(\mathcal{P}_\lambda)$ in a neighborhood of any fixed point $\overline{x} \in \overline{\Omega}_{c^+}$.

- For each $\bar{x} \in \overline{\Omega}_{c^+}$, we have done a local analysis **in a ball, if $\bar{x} \in \overline{\Omega}_{c^+} \cap \Omega$, or in a semiball, if $\bar{x} \in \overline{\Omega}_{c^+} \cap \partial\Omega$.**

Our Strategy

- To prove our results, we have used our Boundary Weak Harnack Inequality to generalize the results in [12], under our setting.

  ✎ **[12] Sirakov, Boyan: A new method of proving a priori bounds for superlinear elliptic PDE, J. Math. Pures Appl. 141, 184-194, 2020.**

- We have showed that it is sufficient **to control the solutions on Ω_{c^+}.**

- Then, we have obtained **a uniform a priori upper bound** for the solutions of $(\mathcal{P}_\lambda)$ in a neighborhood of any fixed point $\bar{x} \in \overline{\Omega}_{c^+}$.

- For each $\bar{x} \in \overline{\Omega}_{c^+}$, we have done a local analysis **in a ball, if $\bar{x} \in \overline{\Omega}_{c^+} \cap \Omega$, or in a semiball, if $\bar{x} \in \overline{\Omega}_{c^+} \cap \partial\Omega$.**

- Similar analyses, based on the use of Harnack type inequalities, **had not been previously performed for the case $\bar{x} \in \partial\Omega$.**

Optimal Assumptions and Possible Generalizations

Remarks

- If global $\mathcal{C}^1$-estimates are proved in the future under more general assumptions, **our method could be adapted** to these situations.

Optimal Assumptions and Possible Generalizations

Remarks

- If global $\mathcal{C}^1$-estimates are proved in the future under more general assumptions, **our method could be adapted** to these situations.
- Considering the lower-order coefficients of $\mathcal{L}$ in $L^q(\Omega)$ **with** $q > n$, **is the optimal Lebesgue integrability** for [3, Theorem 1.1].

Optimal Assumptions and Possible Generalizations

Remarks

- If global $\mathcal{C}^1$-estimates are proved in the future under more general assumptions, **our method could be adapted** to these situations.
- Considering the lower-order coefficients of $\mathcal{L}$ in $\mathcal{L}^q(\Omega)$ **with** $q > n$, **is the optimal Lebesgue integrability** for [3, Theorem 1.1].
- Even the (BPP) **fails, for instance, for** $|b| \in \mathcal{L}^n(\Omega)$**.**

Optimal Assumptions and Possible Generalizations

Remarks

- If global $\mathcal{C}^1$-estimates are proved in the future under more general assumptions, **our method could be adapted** to these situations.
- Considering the lower-order coefficients of $\mathcal{L}$ in $\mathcal{L}^q(\Omega)$ **with** $q > n$, **is the optimal Lebesgue integrability** for [3, Theorem 1.1].
- Even the (BPP) **fails, for instance, for** $|b| \in \mathcal{L}^n(\Omega)$**.**
- Our results **should be true for quasi-linear operators,** if global $\mathcal{C}^1$-estimates were avaliable for the associated Dirichlet problem.

Optimal Assumptions and Possible Generalizations

Remarks

- If global $\mathcal{C}^1$-estimates are proved in the future under more general assumptions, **our method could be adapted** to these situations.
- Considering the lower-order coefficients of $\mathcal{L}$ in $L^q(\Omega)$ **with** $q > n$, **is the optimal Lebesgue integrability** for [3, Theorem 1.1].
- Even the (BPP) **fails, for instance, for** $|b| \in L^n(\Omega)$**.**
- Our results **should be true for quasi-linear operators,** if global $\mathcal{C}^1$-estimates were avaliable for the associated Dirichlet problem.
- [3, Theorem 1.2] should be generalized for divergence form operators with **quadratic growth on the gradient**.

Thank You For Your Attention!

Characterizing the Set of Solutions

- Depending on the parameter $\lambda \in \mathbb{R}$, we study the **existence and multiplicity** of solutions to $(\mathcal{P}_\lambda)$ and obtain a description of the set

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (\mathcal{P}_\lambda)\}.$$

Characterizing the Set of Solutions

- Depending on the parameter $\lambda \in \mathbb{R}$, we study the **existence and multiplicity** of solutions to $(\mathcal{P}_\lambda)$ and obtain a description of the set
$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (\mathcal{P}_\lambda)\}.$$

- As we do **not have global sign conditions**, the previously approaches used in the literature to obtain a priori bounds cannot be applied.

Characterizing the Set of Solutions

- Depending on the parameter $\lambda \in \mathbb{R}$, we study the **existence and multiplicity** of solutions to $(\mathcal{P}_\lambda)$ and obtain a description of the set
$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (\mathcal{P}_\lambda)\}.$$

- As we do **not have global sign conditions**, the previously approaches used in the literature to obtain a priori bounds cannot be applied.

- In fact, the noncoercive case has remained unexplored until very recently.

Characterizing the Set of Solutions

- Depending on the parameter $\lambda \in \mathbb{R}$, we study the **existence and multiplicity** of solutions to $(\mathcal{P}_\lambda)$ and obtain a description of the set

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (\mathcal{P}_\lambda)\}.$$

- As we do **not have global sign conditions**, the previously approaches used in the literature to obtain a priori bounds cannot be applied.

- In fact, the noncoercive case has remained unexplored until very recently.

- We follow the method introduced in [11], which allows to obtain more information about the **qualitative behavior of the solutions.**

✎ **[11] De Coster, C., Fernández, A. J. and Jeanjean, L. A priori bounds and multiplicity of solutions for an indefinite elliptic problem with critical growth, J. Math. Pures Appl. (9), 132, 308-333, 2019.**

Describing Continuums of Solutions

- At first, we show the existence of a continuum of solutions to problem $(\mathcal{P}_\lambda)$, assuming that **problem ($\mathcal{P}_0$), when $\lambda = 0$, has a solution.**

Describing Continuums of Solutions

- At first, we show the existence of a continuum of solutions to problem $(\mathcal{P}_\lambda)$, assuming that **problem $(\mathcal{P}_0)$, when $\lambda = 0$, has a solution.**

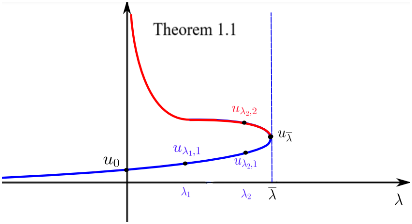
- We consider the cases $c^+(x)u_0 \gneqq 0$ **and** $c^+(x)u_0 \lneqq 0$ **separately,** in [8, Theorem 1.1] and [8, Theorem 1.2], respectively.

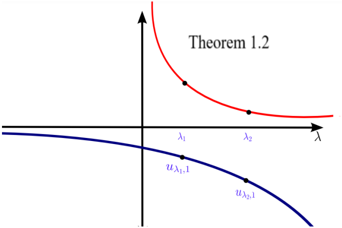
Describing Continuums of Solutions

- At first, we show the existence of a continuum of solutions to problem $(\mathcal{P}_\lambda)$, assuming that **problem $(\mathcal{P}_0)$, when $\lambda = 0$, has a solution.**
- We consider the cases $c^+(x)u_0 \gneqq 0$ **and** $c^+(x)u_0 \lneqq 0$ **separately,** in [8, Theorem 1.1] and [8, Theorem 1.2], respectively.
- In [8, Theorem 1.3], we consider that $(\mathcal{P}_0)$ **does not have a solution, but there exists a supersolution** $\beta_0 \leq 0$ to $(\mathcal{P}_{\lambda_0})$, for some $\lambda_0 > 0$.

Describing Continuums of Solutions

- At first, we show the existence of a continuum of solutions to problem $(\mathcal{P}_\lambda)$, assuming that **problem $(\mathcal{P}_0)$, when $\lambda = 0$, has a solution.**
- We consider the cases $c^+(x)u_0 \gneqq 0$ **and** $c^+(x)u_0 \lneqq 0$ **separately,** in [8, Theorem 1.1] and [8, Theorem 1.2], respectively.
- In [8, Theorem 1.3], we consider that $(\mathcal{P}_0)$ **does not have a solution, but there exists a supersolution** $\beta_0 \leq 0$ to $(\mathcal{P}_{\lambda_0})$, for some $\lambda_0 > 0$.





Describing Continuums of Solutions

- In [8, Theorem 1.5], we also consider the **particular case $h(x) \equiv 0$,** denoting by $\gamma_1 > 0$ is the first eigenvalue of $(\mathcal{P}_{\lambda_1})$.

Describing Continuums of Solutions

- In [8, Theorem 1.5], we also consider the **particular case $h(x) \equiv 0$,** denoting by $\gamma_1 > 0$ is the first eigenvalue of $(\mathcal{P}_{\lambda_1})$.
- Applying the SMP and the Hopf Lemma, we also obtain special cases of [8, Theorems 1.1 and 1.2], **considering the sign of u_0, when $h(x) \gneqq 0$ and $h(x) \lneqq 0$, respectively.**

Describing Continuums of Solutions

- In [8, Theorem 1.5], we also consider the **particular case $h(x) \equiv 0$,** denoting by $\gamma_1 > 0$ is the first eigenvalue of $(\mathcal{P}_{\lambda_1})$.

- Applying the SMP and the Hopf Lemma, we also obtain special cases of [8, Theorems 1.1 and 1.2], **considering the sign of u_0, when $h(x) \gneqq 0$ and $h(x) \lneqq 0$, respectively.**

- For obtaining our results in Theorems 1.1-1.5, it is fundamental to study an **auxiliary fixed point problem via degree theory.**

Describing Continuums of Solutions

- In [8, Theorem 1.5], we also consider the **particular case $h(x) \equiv 0$,** denoting by $\gamma_1 > 0$ is the first eigenvalue of $(\mathcal{P}_{\lambda_1})$.
- Applying the SMP and the Hopf Lemma, we also obtain special cases of [8, Theorems 1.1 and 1.2], **considering the sign of u_0, when $h(x) \gneqq 0$ and $h(x) \lneqq 0$, respectively.**
- For obtaining our results in Theorems 1.1-1.5, it is fundamental to study an **auxiliary fixed point problem via degree theory.**