Sharp non-explicit blow-up profile for a nonlinear heat equation with a gradient term

Mostly Maximum Principle, PUC

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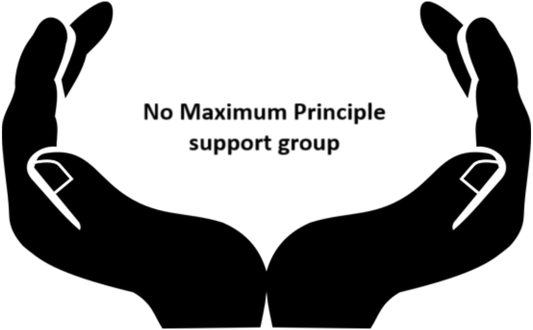
No Maximum Principle
Union


No Maximum Principle
support group

1. Introduction
2. Improvement of the estimate
3. Idea of the proof
   - Parabolic region
   - Outside the parabolic region

Perturbed nonlinear heat equation

$$
\begin{aligned}
u_t &= \Delta u + |u|^{p-1}u + h(u, \nabla u), \\
u(.,0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N),
\end{aligned}
$$

where

- $1 < p$ and if $N \geq 3$, then $(N-2)p < (N+2)$,
- $h \in \mathcal{C}^1(\mathbb{R}^{N+1}), |h(u,v)| \leq C(|u|^\gamma + |v|^{\overline{\gamma}} + 1)$, with $0 \leq \gamma < p, 0 \leq \overline{\gamma} < \frac{2p}{p+1}$ and $C > 0$.

Interpretation (Populations dynamics)

$$u_t = \Delta u + f(u) + h(u, \nabla u).$$

- $u = u(x, t)$ density of the population.
- Δu : diffusion term.
- $f(u)$: birth/death term.
- $h(u, \nabla u)$: predation term (Souplet 1996).

Perturbed nonlinear heat equation

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u + h(u, \nabla u), \\ u(.,0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N), \end{aligned} \tag{1}$$

where

- $1 < p$ and if $N \geq 3$, then $(N-2)p < (N+2)$,
- $h \in \mathcal{C}^1(\mathbb{R}^{N+1}), |h(u,v)| \leq C(|u|^\gamma + |v|^{\overline{\gamma}} + 1)$, with $0 \leq \gamma < p, 0 \leq \overline{\gamma} < \frac{2p}{p+1}$ and $C > 0$.

Cauchy problem : Wellposed in $W^{1,\infty}(\mathbb{R}^\mathbb{N})$ (fixed point argument, see Alfonsi-Weissler (1993), Souplet-Weissler (1999)).

Blow-up solution

Two cases may occur with the maximal solution :

- Either it exists for all $t \geq 0$; we say that the solution is global.
- Or it exists only on some finite interval $[0, T)$ for some $T > 0$. Then, we have

$$||u(t)||_{W^{1,\infty}(\mathbb{R}^N)} \to +\infty \text{ for } t \to T.$$

Definition : Blow-up time : If $T < +\infty$ then we call T the blow-up time.
Definition : Blow-up point : If there is a point $a \in \mathbb{R}^N$ s.t :

$$|u(x_n, t_n)| \to +\infty \text{ for some } (x_n, t_n) \to (a, T),$$

then we say a is a blow-up point.

Earlier work (Explicite profile)

Ebde and Zaag (2011) : there exists a solution s.t

$$||(T-t)^{\frac{1}{p-1}}u(.\sqrt{T-t},t)-f(\frac{.}{\sqrt{|\log(T-t)|}})||_{W^{1,\infty}(\mathbb{R}^N)} \leq \frac{C}{\sqrt{|\log(T-t)|}},$$

Where

$$f(z) = \left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}.$$

Remark : The solution is stable. Thus, we have an open set of initial data, leading to behaviour above mentioned.

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Our goal

We consider the solution constructed by Ebde and Zaag s.t

$$||(T-t)^{\frac{1}{p-1}}u(.\sqrt{T-t},t) - f(\frac{.}{\sqrt{|\log(T-t)|}})||_{W^{1,\infty}(\mathbb{R}^N)} \leq \frac{C}{\sqrt{|\log(T-t)|}}.$$

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Question : Can we get a smaller error ?
Idea : Can we replace f by a sharper profile ?
Remark :

- It can be in fact improved, but we will lose the explicit profile.
- The non-explicit profile is one particular solution of the unperturbed heat equation !

Reformulation of the goal

Idea : We consider two of the constructed solutions u_i for $i \in \{1,2\}$ which blow up in T at the origin.

Similarity variables :

$$y = \frac{x}{\sqrt{T-t}},\ s = -\log(T-t).$$
$$w_i(y,s) = (T-t)^{\frac{1}{p-1}} u_i(x,t).$$

For all $s > s_0 = -\log T$ and $y \in \mathbb{R}^N$

$$w_s = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w + e^{-\frac{p}{p-1}s} h(e^{\frac{1}{p-1}s} w, e^{\frac{p+1}{2(p-1)}s} \nabla w).$$

Reformulation of the goal

We introduce

$$g(y,s) = w_1(y,s) - w_2(y,s).$$

Since w_1 and w_2 have the same profile with error $\frac{1}{\sqrt{s}}$, then we have for free

$$||g||_{W^{1,\infty}(\mathbb{R}^N)} = O(\frac{1}{\sqrt{s}}).$$

Question : Can we improve this ?

Main result

Theorem 1 (B. 2023)

Assume $N \geq 1$ and let be $u_i, i = 1, 2$ two solutions constructed by Ebde and Zaag which blow up at T at the origin. Then for all $(x,t) \in \mathbb{R}^N \times [\max(0, T-1), T)$,

$$|u_1(x,t) - u_2(x,t)| \leq C \min \left\{ \frac{(T-t)^{-\frac{1}{p-1}}}{|\log(T-t)|}, \frac{|x|^{-\frac{2}{p-1}}}{|\log|x||^{\frac{1}{4}-\frac{1}{p-1}}} \right\}.$$

Further improvement

If $N = 1$, we refine the estimate :

Theorem 2 (B. 2023)

Under the same assumptions, there exists $\lambda > 0$ such that :

$$|u_1(x,t) - \lambda^{\frac{2}{p-1}} u_2(\lambda x, T - \lambda^2(T-t))| \\ \leq \max((T-t)^{\beta - \frac{1}{p-1}}, |x|^{2\beta - \frac{1}{p-1}}),$$

where $\beta = \beta(\gamma, \overline{\gamma}) < \min(\frac{1}{2}, \nu)$ and $\nu = \frac{p-\gamma_0}{p-1}$ and $\gamma_0 = \max\{\gamma, \overline{\gamma}(p+1) - p\}$.

Remark :

- Rescaled u_2 is still a solution of the same class of equations.
- We our sharp non-explicit profile ! ! !

Further improvement

If $N = 1$, we refine the estimate :

Theorem 3 (B. '23)

Under the same assumptions, there exists $\lambda > 0$ such that :

$$
\begin{aligned}
&|u_1(x,t) - \lambda^{\frac{2}{p-1}} u_2(\lambda x, \lambda^2 t)| \\
&\leq C \max\left(\frac{(T-t)^{\frac{1}{2}-\frac{1}{p-1}}}{|\log(T-t)|^{\frac{3}{2}}}, \frac{(T-t)^{\nu-\frac{1}{p-1}}}{|\log(T-t)|^{2-\nu}} \exp\left(C\sqrt{-\log(T-t)}\right), \right. \\
&\qquad \frac{|x-a|^{\min(1-\frac{2}{p-1},2\nu)}}{|\log|x-a||^{\min(2-\frac{1}{p-1},\nu)}}, \\
&\qquad \left. \frac{|x-a|^{2\nu-\frac{2}{p-1}}}{|\log|x-a||^{\min(2-\frac{1}{p-1},\nu)}} \exp\left(C\sqrt{-2\log|x-a|}\right) \right),
\end{aligned}
$$

where $\nu = \frac{p-\gamma_0}{p-1}$ and $\gamma_0 = \max\{\gamma, \overline{\gamma}(p+1) - p\}$.

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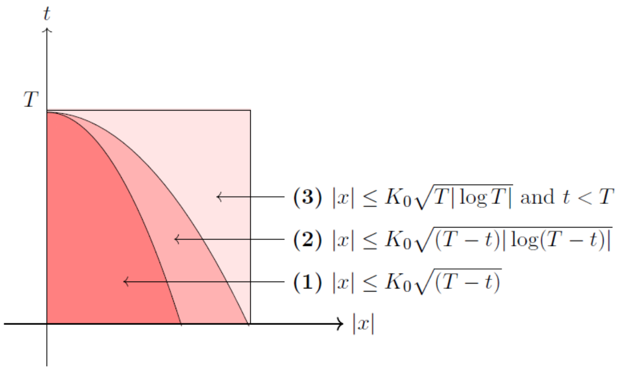
Decomposition of the domain



Region (1) : We have $w \sim f(0) = \kappa$. Thus $u \sim \kappa(T-t)^{-\frac{1}{p-1}}$ with $\kappa = (p-1)^{-\frac{1}{p-1}}$.
Region (2) : We have $w \sim f(\frac{y}{\sqrt{s}})$. Thus $u \sim (T-t)^{-\frac{1}{p-1}} f(\frac{x}{\sqrt{(T-t)|\log(T-t)|}})$.
Region (3) : We have $u' \sim u^p$ (Small diffusion term).

In Parabolic regions ((1) and (2))

- Good results for $h \equiv 0$ (Fermanian and Zaag '00).
- **Goal :** Generalize to the case where $h \not\equiv 0$

The issues :

- No uniform blow-up estimate (in space), because No Liouville Theorem (Merle and Zaag '00) :
  If $h \equiv 0$, in Sobolev subcritical :

$$w \in L^\infty(\mathbb{R}^N, \mathbb{R}) \implies \nabla w \equiv 0.$$

- No parabolic properties : We need a control of the gradient term.

Parabolic regularity

The solution constructed by Ebde and Zaag :

Proposition (B.'23)

There exists $s_1 \geq s_0$ and $M > 0$ such that for all $s \geq s_1$, we have $||w(s)||_{W^{1,\infty}} \leq M$.

L^∞ estimate in the blow-up region

Proposition (B.'23)

For all $K_0 > 0$*, there exist a* $s_1 \geq s_0$ *and a constant* $C(K_0) > 0$ *such that*

$$\forall s \geq s_1,\ \forall |y| \leq K_0\sqrt{s},\ |g(y,s)| \leq \frac{C(K_0)}{s}.$$

For $N = 1$*, there exists* $\sigma_0 \in \mathbb{R}$ *such that*

$$\forall K_0 \in \mathbb{R}^{*+},\ \exists C(K_0) > 0,\ \forall s \geq s_1,\ \forall |y| \leq K_0\sqrt{s},$$

$$|w_1(y,s) - w_2(y,s+\sigma_0)| \leq C(K_0) \max\left(s^{-\frac{3}{2}} e^{-s/2}, s^{\nu-2} e^{-\nu s + C\sqrt{s}}\right).$$

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Outside the parabolic region

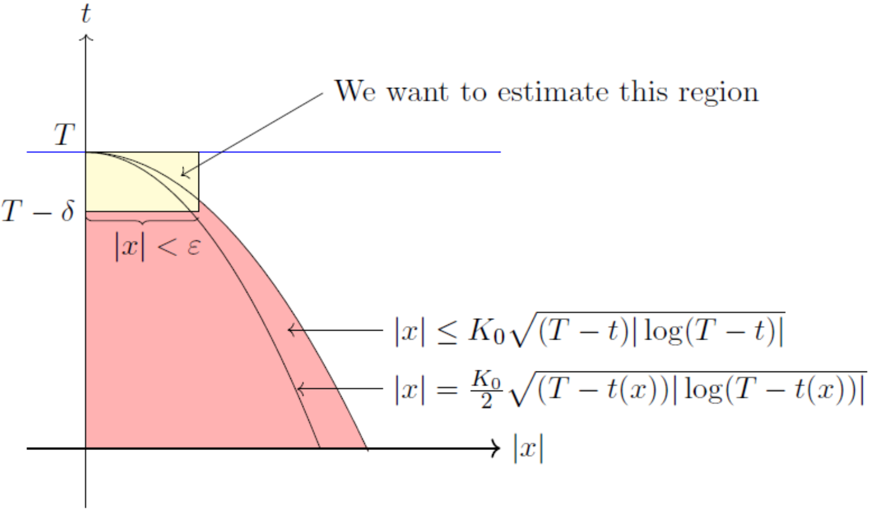
We are in

$$|x| \geq K_0\sqrt{(T-t)|\log(T-t)|)} \text{ with } t \in (T-\delta_0, T).$$

We define for all $|x| \leq \frac{K_0}{2}\sqrt{T|\log T|}$, $t(x) \in [0, T)$ such that

$$|x| = \frac{K_0}{2}\sqrt{(T-t(x))|\log(T-t(x))|}.$$

Outside the parabolic region



Outside the parabolic region

We introduce $\tau(x,t) = \frac{t-t(x)}{T-t(x)}$. Then for all $\tau \in [0,1)$ and $|\xi| \leq \beta(\log(T-t(x)))^{\frac{1}{4}}$, we define

$$v_i(\xi,\tau) = (T-t(x))^{\frac{1}{p-1}} u_1(x+\xi\sqrt{T-t(x)}, t(x)+\tau(T-t(x))).$$

We get

$$\begin{aligned}\partial_\tau v_i = & \Delta v_i + |v_i|^{p-1} v_i \\ & + (T-t(x))^{\frac{p}{p-1}} h_i\left((T-t(x))^{-\frac{1}{p-1}} v_i, (T-t(x))^{-\frac{p+1}{2(p-1)}} \nabla v_i\right).\end{aligned}$$

Estimate outside the parabolic region

$$|v_1(\xi,\tau) - v_2(\xi,\tau)| \leq \frac{C}{|\log(T-t(x))|^{\frac{1}{4}}}.$$

Knowing that

$$\log(T-t(x)) \sim 2\log|x| \text{ and } T-t(x) \sim \frac{2|x|^2}{K_0^2|\log|x||}.$$

Hence, there exists $\epsilon \in (0; K_0\sqrt{T\log T})$ and small δ_0. We have for all $\epsilon \geq |x| \geq K_0\sqrt{(T-t)|\log(T-t)|}$ and $t \in [T-\delta_0, T)$,

$$|u_1(x,t) - u_2(x,t)| \leq \frac{C|x|^{-\frac{2}{p-1}}}{|\log|x||^{\frac{1}{4}-\frac{1}{p-1}}}.$$

Obrigado !

Theorem 4 (B. 2023 **Single-point blow-up and final profile**)

Assume $N \geq 1$ and let be u solution constructed By Ebde and Zaag which blows up at T at the blow up points 0

1. *It holds that 0 is the only blow-up point.*
2. *There exists a final profile u_*, for all $x \in \mathbb{R}^N \backslash \{0\}$. Moreover, there exists $\epsilon > 0$, such that for all $\epsilon \geq |x| > 0$*

$$\left| u_*(x) - \left(\frac{8p|\log|x||}{(p-1)^2|x|^2} \right)^{\frac{1}{p-1}} \right| \leq \frac{C}{|\log|x||^{\frac{1}{4}+\frac{1}{p-1}}},$$

where

$$u_*(x) = \lim_{t \to T} u(x,t).$$