An overview of Moreau's Sweeping Process[1]

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December 21, 2023

An introduction to Nonsmooth Analysis
Center for Mathematical Modeling, Universidad de Chile, Chile.

[1] Partially supported by ANID-Chile under grant Fondecyt Exploración 13220097 and Fondecyt Regular Fondecyt Regular 1220886.

Outline

Aims

The aim of this talk is

1. To introduce a family of differential inclusions, widely used in applications, called Sweeping Processes.
2. To highlight the importance of the sweeping process as a fundamental tool for the mathematical modeling of a wide range of phenomena.
3. To describe some basic tools from convex analysis and differential inclusions used to analyze the sweeping process.
4. To discuss current challenges around the sweeping process and its applications.

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Tools from Convex Analysis[3]

Let H be a separable Hilbert space[2] and $C \subset X$ be a convex set. For $x \in C$ we define the normal cone of C at x as:

$$N(C;x) = \{z \in H \colon \langle z, y - x\rangle \leq 0 \text{ for all } y \in C\}.$$

For example,

$$N(\mathbb{R}_+;x) = \begin{cases} -\mathbb{R}_+ & \text{if } x = 0 \\ \{0\} & \text{if } x > 0 \end{cases}.$$

[2] For example $H = \mathbb{R}^n$ endowed with the usual scalar product.

[3] We refer to [3]: H.H. Bauschke, P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, 2017 for a complete introduction to convex analysis.

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We have that

- if $x \in \operatorname{int}(C)$, then $N(C;x) = \{0\}$.
- if C is the whole space, then $N(C;x) = \{0\}$.

Tools from Convex Analysis

The distance function to a closed and convex set $C \subset H$ is defined by

$$d_C(x) = \inf_{y \in C} \|x - y\|,$$

where $\text{proj}_C(x)$ is the projection of x over the set C:

$$\text{proj}_C(x) = \{y \in C \colon d_C(x) = \|x - y\|\}.$$

Tools from Convex Analysis

The following two relations hold:

- $$x - \operatorname{proj}_C(x) \in N\left(C; \operatorname{proj}_C(x)\right) \quad x \in H.$$
- $$\left(I + N(C;\cdot)\right)^{-1}(x) = \operatorname{proj}_C(x) \quad x \in H.$$

Hausdorff distance

Let $C, D \subset H$ be closed sets, we define the Hausdorff distance between C and D as:

$$\text{Haus}(C,D) = \max\{\sup_{x\in C} d_D(x), \sup_{x\in D} d_C(x)\}.$$

Moreover, it can be proved that

$$\begin{aligned}\text{Haus}(C,D) &= \sup_{x\in H} |d_C(x) - d_D(x)| \\ &= \inf\left\{\eta \geq 0\colon\, C \subset D + \eta\mathbb{B},\, D \subset C + \eta\mathbb{B}\right\}.\end{aligned}$$

Lipschitz moving sets

Let $\mathcal{C} = (C(t))_{t\in[0,T]}$ be a family of closed sets. We say that $\mathcal{C}$ is κ-Lipschitz if

$$\mathrm{Haus}(C(t), C(s)) \leq \kappa|t-s| \quad \text{for all } t, s \in [0,T].$$

Moreover, C is κ-Lipschitz if

$$C(t) \subset C(s) + \kappa|t-s|\mathbb{B} \quad \text{for all } t, s \in [0,T].$$

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Differential inclusions (DIs)

- Let $f \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function.
  A differential equation (DE) is a dynamical system of the form:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0. \end{cases}$$

  Existence theorems: Cauchy-Lipschitz theorem, Peano theorem, etc.

- Let $F \colon [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction.
  A differential inclusion (DI) is a dynamical system of the form:

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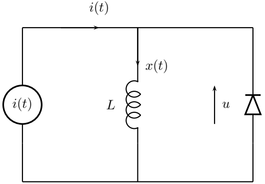
First example: Electrical circuits

Let us consider a circuit with an ideal diode, an inductor and a current source $i(t)$:



If x denotes the current through the inductance, the dynamics is given by

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) - i(t) \\ \mathbb{R}_+ \ni y(t) \perp u(t) \in \mathbb{R}_+. \end{cases} \tag{1}$$

First example: Electrical circuits

The third line in (1) can be written as

$$\mathbb{R}_+ \ni y(t) \perp u(t) \in \mathbb{R}_+ \quad \Leftrightarrow \quad u(t) \in -N\left(\mathbb{R}_+; y(t)\right).$$

Therefore, the system (1) is equivalent to the following differential inclusion:

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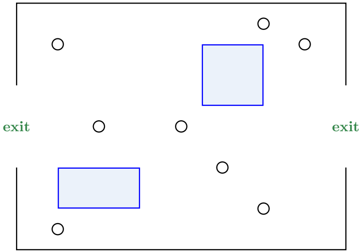
Second example: Crowd motion

Let us consider a model of crowd motion in emergency evacuations:



In this model:

- Consider N persons identified as nonoverlapping rigid disks in $\mathbb{R}^2$.
- Each individual has a spontaneous velocity that he would like to have in absence of other people.

Third example: Crowd motion

The vector of positions $q = (q_1, \ldots, q_N) \in \mathbb{R}^{2N}$ has to belong to the set

$$Q := \{q \in \mathbb{R}^{2N} : D_{ij}(q) \geq 0 \, \forall i \neq j\},$$

where $D_{ij}(q) := \|q_i - q_j\| - 2r$ is the distance between the disk i and j.
If the global spontaneous velocity of the crowd is denoted by

$$V(t,q) = (V_1(t,q_1), \ldots, V_N(t,q_N)) \in \mathbb{R}^{2N},$$

the crowd motion can be described by the following projected differential equation:

$$\frac{dq}{dt} = \mathrm{proj}_{\mathcal{C}_q}(V(t,q)),$$

where

$$\mathcal{C}_q = \{v \in \mathbb{R}^{2N} : \forall i < j, D_{ij}(q) = 0 \Rightarrow \nabla D_{ij}(q) \cdot v \geq 0\},$$

is the set of admissible velocities.
The last projected differential equation is equivalent to:

$$\frac{dq}{dt} \in -N(Q;q) + V(t,q).$$

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Moreau's sweeping process

In all these examples, the motion can be described by the so-called *Sweeeping Processes*:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + f(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases}$$

where $C(t)$ is a closed set for all $t \in [0,T]$ and $f\colon [0,T] \times H \to H$ is an appropriate function.

- The sweeping process was introduced by J.-J Moreau in 1971[4] to model an elastoplastic mechanical system.
- The sweeping process appears in several fields such as nonsmooth electrical circuits, nonsmooth mechanics, crowd motion, hysteresis phenomena, etc.

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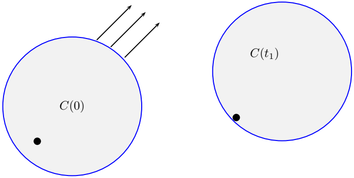
Interpretation of the sweeping process

Consider a large ring that contains a small ball. The ring will start to move at time $t = 0$.
Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.



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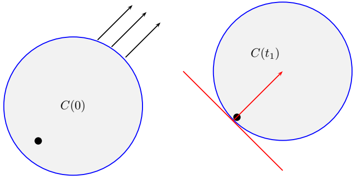
In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.



Interpretation of the sweeping process

Mathematically,

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{3.1}$$

where

- $x(t)$ is the position of the ball at time t.
- $C(t)$ is the moving set (the ring and its interior).
- $N\left(C(t); x(t)\right)$ is some appropriate outward normal cone of $C(t)$ at $x(t) \in C(t)$.

In the general setting, the set $C(t)$ is allowed to change its shape while is moving.

Moreau's sweeping process

Main difficulties:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{3.2}$$

where $C(t)$ is a closed set for all $t \in [0, T]$.

1. The right-hand side of (3.2) is unbounded.
2. The problem (3.2) is a constrained differential inclusion.
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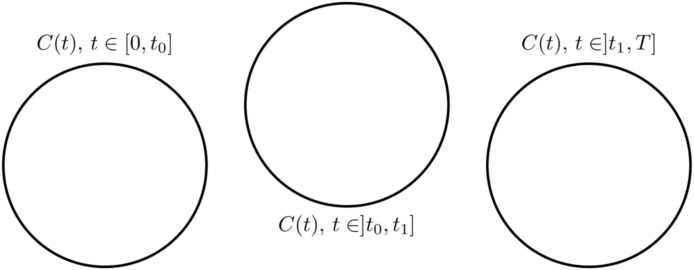

Figure: Sweeping processes without continuous solution. $0 < t_0 < t_1 < T$.

First basic existence result

Theorem (Moreau 1971 [8, 9])

If the sets $(C(t))_{t\geq 0}$ are nonempty, closed and convex with

$$\text{Haus}(C(t), C(s)) = \max\{\sup_{x\in C} d_{C(t)}(x), \sup_{x\in D} d_{C(s)}(x)\} \leq \kappa|t-s|,$$

for some $\kappa \geq 0$. Then, there exists a unique Lipschitz solution of the sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [0,T]. \\ x(0) = x_0 \in C(0). \end{cases} \qquad (SP)$$

Moreover, $\|\dot{x}(t)\| \leq \kappa$ for a.e. $t \in [0,T]$.

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[8] J.J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[9] J.J. Moreau: Rafle par un convexe variable II. Sém. Anal. Convexe Montpellier (1972), Exposé 3.

Proof: Existence

Let $\pi_n = \{0^n, \ldots, t_n^n\}$ be a partition of $[0, T]$. For every $0 \leq i \leq n-1$ we define $I_i^n :=]t_i^n, t_{i+1}^n]$.

Proof: over every I_i^n we use the implicit discretization :

$$\frac{x_{i+1}^n - x_i^n}{t_{i+1}^n - t_i^n} \approx \dot{x}(t) \in -N\left(C(t); x(t)\right) \approx -N\left(C(t_{i+1}^n); x_{i+1}^n\right).$$

This is equivalent to

$$x_i^n \in \left(I + N_{C(t_{i+1}^n)}(\cdot)\right)\left(x_{i+1}^n\right)$$

Hence,

$$\begin{cases} x_{i+1}^n = \text{proj}_{C(t_{i+1}^n)}(x_i^n) & \text{“Catching-up algorithm”}, \\ x_0^n = x_0 \in C(0). \end{cases}$$

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Catching-up algorithm

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Proof: Existence

Thus, the Catching-up enables to define, for every $n \in \mathbb{N}$, the function x_n as: for $t = 0$ $x_n(0) = x_0$, for $t \in I_i^n$ $(0 \leq i \leq n-1)$

$$x_n(t) = x_i^n + \frac{\left(x_{i+1}^n - x_i^n\right)}{t_{i+1}^n - t_i^n}(t - t_i^n).$$

Proof: Existence

It is possible to prove that:

- $$d_{C(t_{i+1}^n)}(x_i^n) \leq \kappa |t_{i+1}^n - t_i^n| \quad \text{for } 0 \leq i \leq n-1.$$
- $x_n(\cdot)$ is κ-Lipschitz continuous over $[0, T]$.

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Proof: Existence

Let $\theta_n \colon [0,T] \to [0,T]$ be the function defined by

$$\theta_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_{i+1}^n & \text{if } t \in I_i^n \text{ for } 0 \le i \le n-1. \end{cases}$$

Then,

$$\begin{cases} \dot{x}_n(t) \in -N\left(C(\theta_n(t)); x_n(\theta_n(t))\right) & \text{a.e. } t \in [0,T], \\ x_n(0) = x_0. \end{cases}$$

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- $(x_n)_n$ is a Cauchy sequence in $C([0, T]; H)$ and, hence, converges to some $x \in C([0, T]; H)$.

Moreover, it is possible to prove that

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Idea: Pass to the limit in the following problem:

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Therefore, the set $K := \{\dot{x}_n(\cdot) \colon n \in \mathbb{N}\}$ is bounded and, thus, relatively weakly compact in $L^1([0,T];H)$.

Finally, by using these theorems, we can pass to the limit in the problem

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First basic existence result

Theorem (Moreau 1971 [8, 9])

If the sets $(C(t))_{t\geq 0}$ are nonempty, closed and convex with

$$\mathrm{Haus}(C(t), C(s)) = \max\{\sup_{x\in C} d_{C(t)}(x), \sup_{x\in D} d_{C(s)}(x)\} \leq \kappa|t-s|,$$

for some $\kappa \geq 0$. Then, there exists a unique Lipschitz solution of the sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [0,T]. \\ x(0) = x_0 \in C(0). \end{cases} \tag{SP}$$

Moreover, $\|\dot{x}(t)\| \leq \kappa$ for a.e. $t \in [0,T]$.

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[8] J.J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[9] J.J. Moreau: Rafle par un convexe variable II. Sém. Anal. Convexe Montpellier (1972), Exposé 3.

Variants and extensions of Moreau's Sweeping Process

- Perturbed state-dependent sweeping processes:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t));x(t)\right) + F(t,x(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

- BV sweeping processes

$$\begin{cases} -dx \in N\left(C(t);x(t)\right) \\ x(0) = x_0 \in C(0), \end{cases}$$

- Second-order sweeping processes:

$$\begin{cases} -\ddot{x}(t) \in N\left(C(t,x(t),\dot{x}(t));\dot{x}(t)\right) + F(t,x(t),\dot{x}(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0, \dot{x}(0) = v_0 \in C(0,x_0,v_0) \end{cases}$$

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I) Moreau's perturbed sweeping process

Theorem (Jourani & Vilches, 2017 [5])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$*
$$\text{Haus}(C(t), C(s)) \leq \kappa|t - s|.$$
- *The family $(C(t))_{t\in[0,T]}$ is positively α-far.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & a.e.\ t \in [0, T]; \\ x(0) = x_0 \in C(0). \end{cases}$$

[5] A. Jourani, E. Vilches, Galerkin-like method and generalized perturbed sweeping process with nonregular sets, *SIAM J. Control Optim.*, 55(4):2412-2436, 2017.

II) State-dependent sweeping process

Theorem (Jourani & Vilches, 2017 [5])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$, $L \in [0, 1[$*

$$\mathrm{Haus}(C(t,x), C(s,y)) \leq \kappa|t-s| + L\|x-y\|.$$

- *The family $\{C(t,x)\colon (t,x) \in [0,T] \times H \times H\}$ is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t)); v(t)\right) + F(t,x(t)) & a.e.\ t \in [0,T]; \\ x(0) = x_0 \in C(0,x_0). \end{cases}$$

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III) Second-order sweeping process

Theorem (Jourani & Vilches, 2017 [5])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$, $L_1 \in [0,1[$ and $L_2 \geq 0$*

$$\text{Haus}(C(t,x,z), C(s,y,w)) \leq \kappa|t-s| + L_1\|x-y\| + L_2\|z-w\|.$$

- *The family $\{C(t,x,y) \colon (t,x,y) \in [0,T] \times H \times H\}$ is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\ddot{x}(t) \in N\left(C(t,x(t),\dot{x}(t)); \dot{x}(t)\right) + F(t,x(t)) & a.e.\ t \in [0,T]; \\ x(0) = x_0, \dot{x}(0) = v_0 \in C(0,x_0,v_0). \end{cases}$$

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IV) Implicit sweeping processes

Let us consider the so-called Implicit Sweeping Process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

where

1. For all $t \in [0,T]$, the set $C(t)$ is a nonempty, closed and convex set.
2. $A, B\colon H \to H$ are bounded, linear and symmetric operators.

This variant was recently introduced by Adly et al ([1, 2]) and then extended by Migórski-Sofonea-Zeng [7] to history-dependent operators.

IV) Implicit sweeping processes

Theorem (Jourani-Vilches, 2019 [6])

Assume that the following conditions hold:

- $A, B\colon H \to H$ *are linear, bounded and symmetric operators such that* $A = P^2$ *for some invertible operator* P.
- *The set-valued map* $C\colon [0,T] \rightrightarrows H$ *is measurable and the function* $t \mapsto d_{C(t)}(0)$ *is integrable.*

Then for any initial point $x_0 \in H$ *there exists a unique absolutely continuous mapping* $x(\cdot; x_0)\colon [0,T] \to H$ *satisfying*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) \text{ a.e. } t \in [0,T], \\ x(0) = x_0. \end{cases}$$

Moreover, the map $x_0 \mapsto x(\cdot; x_0)$ *is locally Lipschitz continuous.*

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[6] A. Jourani, E. Vilches, A differential equation approach to implicit sweeping processes, *J. Differential Equations*, 266(8):5168-5184, 2019.

V) Mirror sweeping processes

Theorem (Chiu, Gutiérrez-Vilches, 2023)

Assume that the following assumptions hold true:

- ϕ *is a mirror map.*
- $x \mapsto f(t,x)$ *is locally Lipschitz and* $t \mapsto f(t,x)$ *is measurable.*
- *For some* $\kappa \geq 0$
$$\text{exc}(C(t), C(s)) \leq \kappa |t - s|.$$
- *The sets* $(C(t))_{t\in[0,T]}$ *are* ρ*-uniformly prox-regular.*

Then, there exists at least one solution of

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); \nabla\phi(x(t))\right) + f(t, x(t)) & a.e.\ t \in [0,T]; \\ x(0) = x_0 \in C(0). \end{cases}$$

VI) Stochastic sweeping processes

Theorem (Garrido-Vilches, 2023)

Assume that the following assumptions hold true:

- $x \mapsto f(t,x)$ *is locally Lipschitz and* $t \mapsto f(t,x)$ *is measurable.*
- $x \mapsto \sigma(t,x)$ *is locally Lipschitz and* $t \mapsto \sigma(t,x)$ *is measurable.*
- *The set-valued map* $t \mapsto C(t)$ *is continuous with convex values and* $\operatorname{int} C(t) \neq \emptyset$.

Then, there exists at least one solution of

$$dX_t \in -N\left(C(t); X_t\right) + f(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Main contributions (non-exhaustive)

1. Well-posedness of solutions:
   - J.-J. Moreau, L. Thibault, M. Bounkhel, V. Recupero, S. Adly, A. Jourani, T. Haddad, S. Zeng, D. Azzam-Laouir, E. Vilches, etc.
2. Stability and asymptotic behavior of sweeping processes:
   - O. Makarenkov, P. Gidoni, G. Colombo, E. Vilches, etc.
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Main challenges (non-exhaustive)

1. Existence of solutions (e.g., numerical algorithms, existence of BV solutions, extensions to non-convex sets, etc.).
2. Stability and asymptotic behavior of sweeping processes and its variants.
3. Deterministic and stochastic optimal control of sweeping processes (optimality conditions in finite and infinite dimensiones, Hamilton-Jacobi Equation, Relaxation, Young Measures, Lyapunov functions, controllability, etc).
4. Applications in crowd motion, nonsmooth mechanics and electrical circuits.

We refer to the the following survey for a detailed overview on the subject:

[4] Brogliato, B. and Tanwani, A., Dynamical Systems Coupled with Monotone Set-Valued Operators: Formalisms, Applications, Well-Posedness, and Stability. *SIAM Rev.*, 62(1):3-129, 2020.

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An overview of Moreau's Sweeping Process[5]

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December 21, 2023
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[5]Partially supported by ANID-Chile under grant Fondecyt Exploración 13220097 and Fondecyt Regular Fondecyt Regular 1220886.