Approximation of stable solutions to a stationay MFG system

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The stationary MFG system

- We consider the stationary MFG system introduced by Bensoussan-Frehse-Yam (2013)

$$\begin{cases} -\Delta u + \frac{1}{2}\left|Du\right|^2 + \lambda u = f(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d, \end{cases} \quad (1)$$

- $\lambda > 0$,
- $f \in C^2(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$,
- $m_0 \in C^{0,\alpha}(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$.

Theorem

There exists a classical solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ *to (1). Furthermore, if* $f' \geq 0$ *then this solution is unique.*

Stable solutions

- If the coupling f is not monotone the solutions to the MFG system are in general not unique.
- For potential MFG systems (in particular for local couplings) Briani and Cardaliaguet introduced Stable Solutions.

Definition (Briani-Cardaliaguet 2018)

Let (u, m) be a classical solution to

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \lambda u = f(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d. \end{cases}$$

We say that (u, m) is *stable* if $(v, \rho) = (0, 0)$ is the unique classical solution to the linearized system

$$\begin{cases} -\Delta v + Du \cdot Dv + \lambda v = f'(m)\rho & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \operatorname{div}(\rho Du) + \lambda \rho = \operatorname{div}(mDv) & \text{in } \mathbb{T}^d. \end{cases}$$

- When the coupling is not monotone, stable solutions are in general not unique. However they have some interesting properties.

Proposition (Briani-Cardaliaguet 2018)

Let (u, m) be a stable solution to

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \lambda u = f(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{2}$$

Then there exists $R > 0$ such that, if $(\tilde{u}, \tilde{m}) \neq (u, m)$ is another classical solution to (2), then

$$\|u - \tilde{u}\|_{W^{1,d}} + \|m - \tilde{m}\|_{L^2} > R.$$

- It is also proved by Briani and Cardaliaguet that the Fictitious Play method converges locally to stable solutions.

- We have the following result on the existence of stable solutions.

Theorem

Every classical solution (u, m) *to*

$$\begin{cases} -\Delta u + \frac{1}{2}\left|Du\right|^2 + \lambda u = f(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{3}$$

is stable if one of the following conditions holds:

1. (Monotonicity of the coupling) $f' \geq 0$.
2. (Large discount factor) $\lambda > \Lambda$, *where* Λ *is a positive constant depending only on the data of the problem.*

- In particular, when the coupling is monotone, the unique solution to (3) is stable.

Abstract reformulation of the MFG system

- Let $T \in \mathcal{L}(C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\alpha}(\mathbb{T}^d), C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d))$ be defined by $T(g,h) = (v,\rho)$ where (v,ρ) solves

$$\begin{cases} -\Delta v + \lambda v = g & \text{in } \mathbb{T}^d, \\ -\Delta \rho + \lambda \rho = h & \text{in } \mathbb{T}^d. \end{cases} \tag{4}$$

  This linear operator is well defined according to the Schauder theory for elliptic equations.

- Let $G \colon C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d) \to C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\alpha}(\mathbb{T}^d)$ be defined by

$$G(v,\rho) := \left(\frac{1}{2} |Dv|^2 - f(\rho), -\operatorname{div}(\rho Du) - \lambda m_0 \right) \tag{5}$$

- Then $(u,m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ is a solution to the MFG system if and only if

$$F(u,m) := (I + T \circ G)(u,m) = 0.$$

More generally we consider Banach spaces X and Z such that

- we have the continuous embeddings

$$C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d) \subset X \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \text{ and } Z \subset L^2(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d).$$

- for every $(v, \rho) \in X$, we have that

$$G(v,\rho) := \left(\frac{1}{2}|Dv|^2 - f(\rho), -\operatorname{div}(\rho Dv) - \lambda m_0\right) \quad \text{belongs to } Z,$$

- for every $(g, h) \in Z$, the problem

$$\begin{cases} -\Delta v + \lambda v = g & \text{in } \mathbb{T}^d, \\ -\Delta \rho + \lambda \rho = h & \text{in } \mathbb{T}^d, \end{cases}$$

admits a unique distributional solution $T(g, h)$ that belongs to X.

Then $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ is a solution to the MFG system if and only if

$$F(u, m) := (I + T \circ G)(u, m) = 0.$$

- Computing formally the differential of G we obtain

$$dG[u,m](v,\rho) = \left(Du \cdot Dv - f'(m)\rho,\ -\operatorname{div}(mDv) - \operatorname{div}(\rho Du)\right).$$

- An element $(v,\rho) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ is then a classical solution to the linearized system

$$\begin{cases} -\Delta v + Du \cdot Dv + \lambda v = f'(m)\rho & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \operatorname{div}(\rho Du) + \lambda \rho = \operatorname{div}(mDv) & \text{in } \mathbb{T}^d, \end{cases} \tag{6}$$

if and only if

$$(I + T \circ dG[u,m])(v,\rho) = 0.$$

- It is possible to prove that, for $(u,m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$, any weak solution $(v,\rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ to (6), i.e. such that

$$\int Dv \cdot D\varphi + Du \cdot Dv\varphi + \lambda v\varphi \, dx = \int f'(m)\rho\varphi \, dx \quad \text{for every } \varphi \in H^1(\mathbb{T}^d),$$
$$\int (-\Delta\psi + Du \cdot D\psi + \lambda\psi)\rho \, dx = -\int mDv \cdot D\psi \, dx \quad \text{for every } \psi \in H^2(\mathbb{T}^d),$$

is a classical solution.

- Therefore (u,m) is a stable solution iff $\ker_{H^1 \times L^2}(I + T \circ dG[u,m]) = \{0\}$.

- The differentiability of G can be rigorously established for Banach spaces of the form

$$X = C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d) \quad Z = C^{0,\beta}(\mathbb{T}^d) \times C^{0,\gamma}(\mathbb{T}^d)$$

and

$$X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \quad Z = L^{p/2}(\mathbb{T}^d) \times W^{-1,r}(\mathbb{T}^d)$$

for appropriate parameters β, γ, p, q, and r.

Theorem (Isomorphism property of stable solutions)

For Banach spaces X and Z as above, and for Y another Banach space such that

$$Y \subset X \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \quad \text{with} \quad Y \subset X \text{ compact.}$$

Assume that the mapping $G\colon X \to Z$ is continuously differentiable and that $T \in \mathcal{L}(Z, Y)$. Then the mapping $F\colon X \to X$ is continuously differentiable with $dF = I + T \circ dG$, and, for every stable solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ to the MFG system, the linear operator $dF[u, m]$ is an isomorphism on X.

Proof.

Let (u, m) be a stable solution to the MFG system. In particular

$$\ker_X(I + T \circ dG[u, m]) \subset \ker_{H^1 \times L^2}(I + T \circ dG[u, m]) = \{0\}.$$

Since $Y \subset X$ is compact and $T \in \mathcal{L}(Z, Y)$ we have that $T \circ dG[u, m] \in \mathcal{L}(X)$ is compact. Using the Fredholm alternative we conclude that

$$R(I + T \circ dG[u, m]) = X.$$

□

- Assume that $d \leq 3$ and $3 < p, q < 6$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Set

$$X = W^{1,p} \times L^q, \quad Y = W^{2,p/2} \times H^1, \quad Z = L^{p/2} \times H^{-1}.$$

  From standard elliptic regularity theory we have $T \in \mathcal{L}(Z, Y)$. From the Rellich-Kondrachov theorem we have $Y \subset X$ with compact embedding.

- Let $0 < \gamma < \beta < \alpha$ and set

$$X = C^{2,\beta} \times C^{2,\gamma}, \quad Y = C^{2,\alpha} \times C^{2,\beta} \quad \text{and} \quad Z = C^{0,\alpha} \times C^{0,\beta}.$$

  From the Schauder theory for elliptic equations we have $T \in \mathcal{L}(Z, Y)$ and from properties of Hölder spaces we have $Y \subset X$ with compact embedding.

In particular we have the following results.

Corollary

Let $p, q > d$ and such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and let $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to the MFG system. Then $dF[u, m] = (I + T \circ dG[u, m])$ is an isomorphism on $W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$.

Corollary

Let $\gamma < \beta < \alpha$ and let $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to the MFG system. Then $dF[u, m] = (I + T \circ dG[u, m])$ is an isomorphism on $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$.

Application to FEM approximations of stable solutions

Theorem (Brezzi-Rappaz-Raviart 1980)

Let X and Z be Banach spaces, let $T, T_h \in \mathcal{L}(Z, X)$ and $G\colon X \to Z$ be C^1 and assume that dG is locally Lipschitz continuous. Set $F = I + T \circ G$ and $F_h = I + T_h \circ G$. Assume that $\bar{x} \in X$ is such that $F(\bar{x}) = 0$ and $dF[\bar{x}]$ is an isomorphism on X. Suppose also that

$$\lim_{h \to 0} \|T - T_h\|_{\mathcal{L}(Z,X)} = 0.$$

Then there exists $h_0 > 0$ such that for every $0 < h \leq h_0$ there exists a $\bar{x}_h \in X$ such that

$$F_h(\bar{x}_h) = 0.$$

Moreover we have the error estimate

$$\|\bar{x} - \bar{x}_h\|_X \leq C \, \|(T - T_h)G(\bar{x})\|_X$$

for some positive constant C and for every $0 < h \leq h_0$ this solution is unique in a neighborhood of $\bar{x}$ which is independent of h.

- Fix $d \leq 3$. For every $h > 0$ we consider T_h the $\mathbb{P}^1$ finite elements approximation of T for an appropriate mesh of size h.
- Let $3 < p, q < 6$ and consider

$$X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \quad \text{and} \quad Z = L^{p/2}(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d).$$

- Then (u_h, m_h) is a solution to the discrete MFG system if

$$F_h(u_h, m_h) := (I + T_h \circ G)(u_h, m_h) = 0.$$

- Using standard FEM estimates and interpolation theory it is possible to prove that

$$\|T - T_h\|_{\mathcal{L}(Z,X)} \leq Ch^{(p-3)/p}.$$

Applying the previous theorem to the MFG system we obtain the following.

Theorem

Let $3 < p, q < 6$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Let also (u, m) be stable solution to the MFG system. Then there exists a neighborhood $\mathcal{O}$ of (u, m) in $W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ and $h_0 > 0$ such that for every $0 < h \leq h_0$ there exists $(u_h, m_h) \in \mathcal{O}$, unique in $\mathcal{O}$, such that

$$F_h(u_h, m_h) = 0$$

and we have the error estimate

$$\|u - u_h\|_{W^{1,p}} + \|m - m_h\|_{L^q} \leq Ch^{(p-3)/p}.$$

- Finite element aproximations for this stationary MFG system with monotone coupling was considered in Osborne-Smears (2023).

Application to the classical Newton method

Theorem (Classical Newton method in infinite dimensions)

Let X and Y be Banach spaces and $F\colon X \to Y$ be continuously differentiable. Let $\bar{x} \in X$ be such that $F(\bar{x}) = 0$ and $dF[\bar{x}] \in \mathcal{L}(X, Y)$ is an isomorphism. Then there exists a neighborhood $\mathcal{O}$ of $\bar{x}$ in X such that the sequence defined by

$$\begin{cases} x_0 \in \mathcal{O}, \\ F(x_k) + dF[x_k](x_{k+1} - x_k) = 0, \end{cases} \tag{7}$$

converges superlinearly to $\bar{x}$. Furthermore, if dF is locally Lipschitz continuous in $\mathcal{L}(X, Y)$, then the convergence is quadratic.

Theorem

Let $0 < \gamma < \beta < \alpha$ and $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to the MFG system. Then there exists a neighborhood $\mathcal{O}$ of of (u, m) in $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$ such that if $(u_0, m_0) \in \mathcal{O}$ then the sequence (u_k, m_k) generated by Newton's method, i.e. the sequence defined by

$$(u_{k+1}, m_{k+1}) + T\left(G(u_k, m_k) + dG[u_k, m_k](u_{k+1} - u_k, m_{k+1} - m_k)\right) = 0, \qquad (8)$$

converges super-linearly to (u, m) in $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$. Furthermore, if we also assume $f \in C^{2,1}_{\mathrm{loc}}(\mathbb{R})$, then the convergence is quadratic.

- Convergence rates for parabolic MFG systems with monotone couplings were obtained in Camilli-Tang (2023).