A high-order scheme for mean field games

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Overview

Introduction

Motivations

- MFGs characterize Nash equilibria of stochastic differentiable games with infinitely many players; in some cases such equilibria are given by a system of PDEs (a forward FP equation and a backward HJB equation).
- There are several methods based on the finite difference scheme by Chang and Cooper[1] which require a parabolic CFL in order to be explicit and stable.
- Only few works deal with high-order numerical schemes for MFG systems, that are finite-difference based[2] or that imply high-order space-time finite elements to approximate variational MFGs[3].
- Our purpose is to provide a new scheme which is explicit, conservative, consistent, convergent, high-order and stable without a CFL condition for the FP equation.
- We introduce a high-order SL method for HJB equation and couple it with our scheme for FP to solve the MFG problem.

[1] Chang and Cooper, "A practical difference scheme for Fokker-Planck equations".

[2] Li, Fan, and Ying, "A simple multiscale method for mean field games"; Popov and Tomov, "Central schemes for mean field games".

[3] Fu et al., "High order computation of optimal transport, mean field planning, and potential mean field games".

FP equation
The equation

We consider here the linear Fokker-Planck equation

$$\begin{cases} \partial_t m - \frac{\sigma^2}{2}\Delta m + \operatorname{div}(\mu m) = 0 & \text{in } (0,T] \times \mathbb{R}^d, \\ m(0,\cdot) = \overline{m}_0 & \text{in } \{0\} \times \mathbb{R}^d. \end{cases} \tag{\textbf{FP}}$$

(H1) We assume that:

(i) $\overline{m}_0 \in C_0^0(\mathbb{R}^d)$ has compact support, $\overline{m}_0 \geq 0$, and $\int_{\mathbb{R}^d} \overline{m}_0(x)\mathrm{d}x = 1$.

(ii) μ is bounded, $\mu \in C([0,T] \times \mathbb{R}^d)$, and there exists $C_b > 0$ such that

$$|\mu(s,x) - \mu(t,y)| \leq C_b|x-y| \quad \text{for } t \in [0,T] \text{ and } x,\, y \in \mathbb{R}^d.$$

FP equation

The equation - Existence of solution

Under the assumptions stated in **(H1)** the following hold[4,5]:

- Equation (**FP**) admits a unique classical solution $m^* \in C^{1,2}([0,T] \times \mathbb{R}^d)$;
- $m^* \geq 0$;
- $\int_{\mathbb{R}^d} m^*(t,x)\mathrm{d}x = 1$ for all $t \in [0,T]$;
- m^* is the unique solution in $L^2([0,T] \times \mathbb{R}^d)$ to (**FP**) in the distributional sense.

For each $\phi \in C_0(\mathbb{R}^d)$, in $[t_1, t_2] \subset [0,T]$ we have

$$\int_{\mathbb{R}^d} \phi(x) m(t_2, x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\left(\phi(X(t_2; t_1, x)) m(t_1, x)\right) \mathrm{d}x. \tag{1}$$

[4] Bogachev et al., Fokker-Planck-Kolmogorov equations, Theorem 6.6.1, Chapter 9.1.

[5] Figalli, "Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients", Proposition 4.4, Theorem 4.3.

FP equation

Discretization of the characteristics

$X(s;t,x)$ is defined as the unique solution to

$$\begin{aligned} \mathrm{d}X(s) &= \mu(s, X(s))\mathrm{d}s + \sigma \mathrm{d}W(s) \quad \text{for } s > t \\ X(t) &= x. \end{aligned}$$

We recall that using the Crank-Nicolson approximation, $y_{t_1,x}$ is the solution of

$$y_{t_1,x}(t_2) = x + \frac{(t_2 - t_1)}{2}\left(\mu(t_1, x) + \mu(t_2, y_{t_1,x}(t_2))\right) + \sqrt{(t_2 - t_1)}\sigma\xi, \quad (2)$$

where ξ is an $\mathbb{R}^d$-valued random vector with i.i.d. components such that

$$\mathbb{P}(\xi_i = 0) = 2/3, \quad \mathbb{P}(\xi_i = \pm\sqrt{3}) = 1/6 \quad \text{for} \quad i = 1, \ldots, d. \quad (3)$$

LG scheme

Derivation of the scheme

Using the Crank-Nicolson approximation for the characteristics, there exists $C > 0$ such that for ϕ smooth enough

$$\left| \mathbb{E}\left(\phi(X(t_2; t_1, x))\right) - \mathbb{E}\left(\phi(y_{t_1,x}(t_2))\right) \right| \leq C(t_2 - t_1)^3. \tag{4}$$

- Let $\{e_\ell \,|\, \ell = 1, \ldots, 3^d\} \subset \mathbb{R}^d$ be the set of possible realizations of ξ;
- set $\omega_\ell = \mathbb{P}(\xi = e_\ell)$;
- denote $y_k^\ell(x)$ as the unique solution to the Crank-Nicolson fixed point for the characteristics in (2), for $\xi = e_\ell$, $\ell = 1, \ldots, 3^d$.

LG scheme

Derivation of the scheme

- Let $\mathcal{I} = \{1, \dots, 3^d\}$;
- choose $N_{\Delta t} \in \mathbb{N}$;
- set $\Delta t = T/N_{\Delta t}$, $t_k = k\Delta t$;

the representation formula in $[t_k, t_{k+1}]$

$$\int_{\mathbb{R}^d} \phi(x) m(t_{k+1}, x) \mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\left(\phi(X(t_{k+1}; t_k, x)) m(t_k, x)\right) \mathrm{d}x$$

can be approximated as

$$\int_{\mathbb{R}^d} \phi(x) m(t_{k+1}, x) \mathrm{d}x = \sum_{\ell \in \mathcal{I}} \omega_\ell \int_{\mathbb{R}^d} \phi(y_k^\ell(x)) m(t_k, x) \mathrm{d}x + O((\Delta t)^3). \qquad (5)$$

LG scheme

Derivation of the scheme

- For fixed Δt, the boundedness of μ implies that there exists $L_{\Delta t} \sim C/\sqrt{\Delta t}$ such that for any $k = 0, \ldots, N_T$

$$\mathrm{supp}(m_\Delta(t_k)) \subset [-L_{\Delta t}, L_{\Delta t}]^d =: \mathcal{O}_\Delta;$$

- choose a $\Delta x > 0$ and introduce a structured mesh $\mathcal{G}_\Delta := \{(t_k, x_i)\}$;
- consider odd[6] symmetric Lagrange interpolation basis functions $\{\beta_i\}_{i\in\mathbb{Z}^d}$ (stable), which are tensor product of the one-dimensional reference function $\hat{\beta}(\xi)$;
- define a projection on the space spanned by the basis $\{\beta_i\}_i$,

$$m_\Delta(t_k, \cdot) = \sum_{i\in\mathcal{I}_{\Delta x}} m_{k,i}\beta_i(x).$$

[6] Ferretti, "On the relationship between semi-Lagrangian and Lagrange-Galerkin schemes"; Ferretti and Mehrenberger, "Stability of semi-Lagrangian schemes of arbitrary odd degree under constant and variable advection speed".

LG scheme

Derivation of the scheme

Recall the formula

$$\int_{\mathbb{R}^d} \phi(x) m(t_{k+1}, x) \mathrm{d}x = \sum_{\ell \in \mathcal{I}} \omega_\ell \int_{\mathbb{R}^d} \phi(y_k^\ell(x)) m(t_k, x) \mathrm{d}x + O((\Delta t)^3),$$

- we set $\phi = \beta_i$;
- we substitute $m(t_k, \cdot)$ with $m_\Delta(t_k, \cdot) = \sum_{j \in \mathcal{I}_{\Delta x}} m_{k,j} \beta_j^q(x)$;

$$\int_{\mathbb{R}^d} \phi(x) m(t_{k+1}, x) \mathrm{d}x \approx \sum_{j \in \mathcal{I}_{\Delta x}} m_{k+1,j} \int_{\mathcal{O}_\Delta} \beta_i(x) \beta_j(x) \mathrm{d}x$$

$$\int_{\mathbb{R}^d} \phi(y_k^\ell(x)) m(t_k, x) \mathrm{d}x \approx \sum_{j \in \mathcal{I}_{\Delta x}} m_{k,j} \int_{\mathcal{O}_\Delta} \beta_i(y_k^\ell(x)) \beta_j(x) \mathrm{d}x.$$

LG scheme
Derivation of the scheme

We define the mass matrix D and the matrix B_k^ℓ,

$$D_{i,j} = \int_{\mathcal{O}_\Delta} \beta_i(x)\beta_j(x)\mathrm{d}x, \quad b_{i,j,k}^\ell = \int_{\mathcal{O}_\Delta} \beta_i(y_k^\ell(x))\beta_j(x)\mathrm{d}x$$

for $(i,j) \in \mathcal{I}_{\Delta x} \times \mathcal{I}_{\Delta x}$.
The scheme is the following: find $\{m_{k,j} \in \mathbb{R}, j \in \mathcal{I}_{\Delta x}, k \in \mathcal{I}_{\Delta t}^*\}$ such that for all $i \in \mathcal{I}_{\Delta x}$ and $k \in \mathcal{I}_{\Delta t}^*$

$$\begin{aligned} \sum_{j \in \mathcal{I}_{\Delta x}} D_{i,j} m_{k+1,j} &= \sum_{j \in \mathcal{I}_{\Delta x}} \sum_{\ell \in \mathcal{I}} \omega_\ell b_{k,i,j}^\ell m_{k,j}, \\ \sum_{j \in \mathcal{I}_{\Delta x}} D_{i,j} m_{0,j} &= \int_{\mathcal{O}_\Delta} m_0(x)\beta_i(x)\mathrm{d}x. \end{aligned} \tag{6}$$

LG scheme

Properties of the exactly integrated scheme

For fixed $\Delta t > 0$ and $\Delta x > 0$ we proved the following properties:

- [Well-posedness] given $m_0 \in \mathbb{R}^{(N_{\Delta x})^d}$, there exists a unique solution $m_k \in \mathbb{R}^{(N_{\Delta x})^d}$ to (6), for $k = 0, \dots, N_{\Delta t} - 1$;
- [Initial condition] $\|\overline{m}_0 - m_\Delta(0,\cdot)\|_{L^2(\mathbb{R}^d)} = O((\Delta x)^{q+1})$ if $\overline{m}_0 \in H^{q+1}(\mathbb{R}^d)$;
- [Mass conservation] $\int_{\mathcal{O}_\Delta} m_\Delta(t_k, x)\mathrm{d}x = 1$ for $k \in \mathcal{I}_{\Delta t}$;
- [L^2-stability] if μ is differentiable w.r.t. x, then $\max_{k\in\mathcal{I}_{\Delta t}} \|m_\Delta(t_k,\cdot)\|_{L^2}$ is uniformly bounded with respect to Δ for Δt small enough.

LG scheme

Properties of the exactly integrated scheme

Proposition (Consistency)

Let $\phi \in C_0^\infty(\mathbb{R}^d)$, then for any $k \in \mathcal{I}_{\Delta t}^*$ and for any $(\Delta t, \Delta x) \in (0, +\infty)^2$, the scheme is consistent.

Proposition (Convergence)

Assume that $\overline{m}_0 \in H^{q+1}(\mathbb{R}^d)$, that **(H1)** *holds, and that b is differentiable w.r.t. x. Consider a sequence $(\Delta_n)_{n\in\mathbb{N}} = ((\Delta t_n, \Delta x_n))_{n\in\mathbb{N}} \subseteq (0,\infty)^2$ such that, as $n \to \infty$, $(\Delta t_n, \Delta x_n) \to (0,0)$ and $(\Delta x_n)^{q+1}/\Delta t_n \to 0$. Setting $m^n := m_{\Delta_n}$, as $n \to \infty$ we have that $(m^n)_{n\in\mathbb{N}}$ converges to m in $C([0,T]; \mathcal{D}'(\mathbb{R}^d))$ and weakly in $L^2\left([0,T] \times \mathbb{R}^d\right)$, where m is the unique classical solution to* **(FP)**.

LG scheme

Implementation of the method in dimension one

In one spatial dimension, to get a method of order two we use Simpson quadrature rule on each element $[x_j, x_j + 2\Delta x]$, with $j = 2m$ and $m \in \mathbb{Z}$, and cubic symmetric Lagrange interpolation basis functions $(\beta_i^3(x))$.
Each basis $\beta_j^3(x)$ has support in $[x_{j-2}, x_{j+2}]$, so the elements of matrices D and B_k^ℓ are approximated as follows:

$$D_{i,j} \simeq \frac{2\Delta x}{3}\delta_{i,j}, \quad b_{k,i,j}^\ell \simeq \frac{2\Delta x}{3}\beta_i^3(y_k^\ell(x_j)), \quad \forall i, j, \in \mathcal{I}_{\Delta x}.$$

Simplifying $\frac{2\Delta x}{3}$ in both approximations, the method results in

$$m_{k+1} = \sum_{\ell \in \mathcal{I}} \omega_\ell \widetilde{B}_k^\ell m_k \tag{7}$$

with $\widetilde{B}_k^\ell : \mathbb{Z}^2 \to \mathbb{R}$ linear operator with $\widetilde{b}_{k,i,j}^\ell$ defined by

$$\widetilde{b}_{k,i,j}^\ell = \beta_i^3(y_k^\ell(x_j)).$$

Application to MFGs

The MFGs problem

Let us consider a Mean Field Game (MFG) problem of the form:

$$
\begin{cases}
-\partial_t v - \frac{\sigma^2}{2}\Delta v + \frac{1}{2}|\nabla v|^2 = F(x, m(t)), & (t,x) \in [0,T) \times \mathbb{R}^d, \\
\partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\left(\nabla v m\right) = 0 & (t,x) \in (0,T] \times \mathbb{R}^d, \\
v(t,x) = G(x, m(T)) & x \in \mathbb{R}^d \\
m(0) = m_0, & m_0 \in \mathcal{P}_1(\mathbb{R}^d),
\end{cases} \qquad (\mathbf{MFG})
$$

where $\sigma \in \mathbb{R}$, $F : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$, $G : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$.

- The first equation in $(\mathbf{MFG})$ is a Hamilton-Jacobi-Bellman (HJB) equation with cost depending on m;
- the distribution m is the solution of the second equation, which is a Fokker-Planck with drift given by ∇v.

Application to MFGs

The MFGs problem

(H2) We assume that:

(i) m_0^* is nonnegative, Hölder continuous, and $\int_{\mathcal{R}^d} \overline{m}_0(x)\mathrm{d}x = 1$.

(ii) F and G are bounded and Lipschitz continuous. Moreover, for every $m \in \mathcal{P}_1(\mathbb{R}^d)$, $F(\cdot, m)$ is of class C^2 and

$$\sup_{x\in\mathbb{R}^d, m\in\mathcal{P}_1(\mathbb{R}^d)} \left\{ \|DF(x,m)\|_\infty + \|D^2F(x,m)\|_\infty \right\} < \infty.$$

Under **(H2)** system (**MFG**) admits at least one classical solution[7]. Moreover, if the coupling terms F and G satisfy a monotonicity condition with respect to m, then the classical solution is unique[8].

[7] Cardaliaguet, "Notes on Mean Field Games: from P.-L. Lions' lectures at Collège de France", Theorem 3.1.

[8] Lasry and Lions, "Mean field games", Theorem 2.4.

Application to MFGs

Semi-Lagrangian scheme for the HJB equation

We consider a *Semi-Lagrangian* (SL) approximation of the HJB equation[9]. For a given $m \in C([0,T],\mathcal{P}_1)$, we define the operator

$$S_\Delta[m](f,k,i) := \inf_{\alpha\in\mathbb{R}^d}\left[\sum_{\ell\in\mathcal{I}} \omega_\ell \left(I[f](x_i+\Delta t\alpha+\sqrt{\Delta t}\sigma e_\ell)+ \frac{\Delta t}{2}F(x_i+\Delta t\alpha+\sqrt{\Delta t}\sigma e_\ell, m(t_k))\right)+\frac{\Delta t}{2}|\alpha|^2\right]+\frac{\Delta t}{2}F(x_i,m(t_{k+1})).$$

The scheme for the HJB equation is: find a sequence $\{v_{k,i}\in\mathbb{R}, k\in\mathcal{I}_{\Delta t}, i\in\mathcal{I}_{\Delta x}\}$ such that, for $i\in\mathbb{Z}^d, k\in\mathcal{I}^*_{\Delta t}$,

$$\begin{cases} v_{k,i} = S_\Delta[m](v_{k+1},k,i), \\ v_{N_{\Delta t},i} = G(x_i, m(t_{N_{\Delta t}})). \end{cases} \tag{8}$$

[9] Bonaventura et al., “Second order fully semi-Lagrangian discretizations of advection-diffusion-reaction systems”.

Application to MFGs

The scheme for MFG

We propose the following scheme for (**MFG**): find a fixed point $\{(v_{k,i}, m_{k,j}) \in \mathbb{R}^2, k \in \mathcal{I}_{\Delta t}, j \in \mathcal{I}_{\Delta x}\}$ of the discrete system

$$\begin{cases} v_{k,i} = S_\Delta[m](v_{k+1}, k, i), \\ v_{N_{\Delta t},i} = G(x_i, m) \\ \displaystyle\sum_{j\in\mathcal{I}_{\Delta x}} D_{i,j} m_{k+1,j} = \sum_{j\in\mathcal{I}_{\Delta x}} m_{k,j} \sum_{\ell\in\mathcal{I}} \omega_\ell b^\ell_{i,j,k}[v] \\ \displaystyle\sum_{j\in\mathcal{I}_{\Delta x}} D_{i,j} m_{0,i} = \int_{\mathcal{O}_\Delta} m_0(x)\beta_i(x)\mathrm{d}x, \end{cases} \tag{9}$$

for all $i \in \mathcal{I}_{\Delta x}$ and $k \in \mathcal{I}^*_{\Delta t}$, where $b^\ell_{i,j,k}[v] = \int_{\mathcal{O}_\Delta} \beta_i(y^\ell_k[v](x))\beta_j(x)\mathrm{d}x$.

Numerical results

Heuristic truncation error

For the exactly integrated scheme (6), the local truncation error is given by the contributions of the Crank-Nicolson estimate

$$\left|\mathbb{E}\left(\phi(X(t_{k+1};t_k,x))\right) - \mathbb{E}\left(\phi(y_{t_k,x}(t_{k+1}))\right) = O((\Delta t)^3)\right)$$

and the interpolation error

$$|f(x) - I[f](x)| = O((\Delta x)^{q+1}),$$

which yield a global truncation error of order $(\Delta x)^{q+1}/\Delta t + (\Delta t)^2$. Taking $\Delta t = O((\Delta x)^{(q+1)/3})$ the order of consistency is maximized[10], obtaining an order of convergence of $2(q+1)/3$.

[10] Ferretti, "A technique for high-order treatment of diffusion terms in semi-Lagrangian schemes".

Numerical results

Non local MFG with analytical solution

Consider the Mean Field Game expressed by the following equations:

$$
\begin{cases}
-\partial_t v - \frac{\sigma^2}{2}\Delta v + \frac{1}{2}\left|\nabla v\right|^2 = \frac{1}{2}\left(x - \int_{\mathcal{O}} \varepsilon \mathrm{d}m_t\left(\varepsilon\right)\right)^2 & (t,x) \in (0,T) \times \mathcal{O}_\Delta, \\
\partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\left((\nabla v)m\right) = 0 & (t,x) \in (0,T) \times \mathcal{O}_\Delta, \\
v(t,x) = v_{ex}(t,x), \quad v(t,x) = v_{ex}(t,x) & (t,x) \in (0,T) \times \partial\mathcal{O}_\Delta, \\
v(T,x) = 0 & x \in \mathcal{O}_\Delta, \\
m(t,x) = m_{ex}(t,x), \quad m(t,x) = m_{ex}(t,x) & (t,x) \in (0,T) \times \partial\mathcal{O}_\Delta, \\
m(0,x) = m_0 & x \in \mathcal{O}_\Delta,
\end{cases}
$$

with $T = 0.25$, $\mathcal{O}_\Delta = (-2,2)$. We approximated the gradient of the value function using a fourth-order finite difference scheme and we performed our tests using $\Delta t = O\left((\Delta x)^{4/3}\right)$, $\Delta t = O\left(\Delta x\right)$, and $\Delta t = O\left((\Delta x)^2\right)$.

Numerical results

Non local MFG with analytical solution - Errors on the HJB

Δx	Errors for the approximation of $v^*(0,\cdot)$			
	E_∞	E_2	p_∞	p_2
$2.00 \cdot 10^{-1}$	$1.68 \cdot 10^{-4}$	$1.70 \cdot 10^{-4}$	-	-
$1.00 \cdot 10^{-1}$	$3.56 \cdot 10^{-5}$	$3.48 \cdot 10^{-5}$	2.24	2.29
$5.00 \cdot 10^{-2}$	$5.86 \cdot 10^{-6}$	$5.75 \cdot 10^{-6}$	2.60	2.60
$2.50 \cdot 10^{-2}$	$1.06 \cdot 10^{-6}$	$1.04 \cdot 10^{-6}$	2.47	2.47

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = (\Delta x)^{4/3}/4$

Numerical results

Non local MFG with analytical solution - Errors on the FP

Δx	Errors for the approximation of $m^*(T,\cdot)$				
	E_∞	E_2	p_∞	p_2	positivity error
$2.00 \cdot 10^{-1}$	$8.81 \cdot 10^{-3}$	$1.01 \cdot 10^{-2}$	-	-	$-3.51 \cdot 10^{-4}$
$1.00 \cdot 10^{-1}$	$3.06 \cdot 10^{-3}$	$2.53 \cdot 10^{-3}$	1.53	2.00	$-9.45 \cdot 10^{-9}$
$5.00 \cdot 10^{-2}$	$8.01 \cdot 10^{-4}$	$5.56 \cdot 10^{-4}$	1.93	2.19	0
$2.50 \cdot 10^{-2}$	$1.81 \cdot 10^{-4}$	$1.14 \cdot 10^{-4}$	2.15	2.29	0

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = (\Delta x)^{4/3}/4$

Numerical results

Non local MFG with analytical solution - Errors on the HJB

Δx	Errors for the approximation of $v^*(0,\cdot)$			
	E_∞	E_2	p_∞	p_2
$2.00 \cdot 10^{-1}$	$2.72 \cdot 10^{-4}$	$2.56 \cdot 10^{-4}$	-	-
$1.00 \cdot 10^{-1}$	$7.62 \cdot 10^{-5}$	$6.72 \cdot 10^{-5}$	1.84	1.93
$5.00 \cdot 10^{-2}$	$1.61 \cdot 10^{-5}$	$1.44 \cdot 10^{-5}$	2.24	2.22
$2.50 \cdot 10^{-2}$	$3.69 \cdot 10^{-6}$	$3.59 \cdot 10^{-6}$	2.13	2.00

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = \Delta x/4$.

Numerical results

Non local MFG with analytical solution - Errors on the FP

Δx	Errors for the approximation of $m^*(T,\cdot)$				
	E_∞	E_2	p_∞	p_2	positivity error
$2.00 \cdot 10^{-1}$	$5.93 \cdot 10^{-3}$	$7.01 \cdot 10^{-3}$	-	-	$-8.18 \cdot 10^{-5}$
$1.00 \cdot 10^{-1}$	$2.63 \cdot 10^{-3}$	$2.17 \cdot 10^{-3}$	1.17	1.69	$-3.58 \cdot 10^{-10}$
$5.00 \cdot 10^{-2}$	$1.23 \cdot 10^{-3}$	$4.80 \cdot 10^{-4}$	1.10	2.18	0
$2.50 \cdot 10^{-2}$	$3.39 \cdot 10^{-4}$	$9.61 \cdot 10^{-5}$	1.86	2.32	0

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = \Delta x/4$.

Numerical results

Non local MFG with analytical solution - Errors on the HJB

Δx	Errors for the approximation of $v^*(0,\cdot)$			
	E_∞	E_2	p_∞	p_2
$2.00 \cdot 10^{-1}$	$1.98 \cdot 10^{-4}$	$1.87 \cdot 10^{-4}$	-	-
$1.00 \cdot 10^{-1}$	$2.84 \cdot 10^{-5}$	$2.86 \cdot 10^{-5}$	2.80	2.71
$5.00 \cdot 10^{-2}$	$3.41 \cdot 10^{-6}$	$3.94 \cdot 10^{-5}$	3.06	2.86
$2.50 \cdot 10^{-2}$	$4.56 \cdot 10^{-7}$	$5.08 \cdot 10^{-6}$	2.90	2.96

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = (\Delta x)^2$.

Numerical results

Non local MFG with analytical solution - Errors on the FP

Δx	Errors for the approximation of $m^*(T,\cdot)$				
	E_∞	E_2	p_∞	p_2	positivity error
$2.00 \cdot 10^{-1}$	$6.60 \cdot 10^{-3}$	$7.63 \cdot 10^{-3}$	-	-	$-1.01 \cdot 10^{-4}$
$1.00 \cdot 10^{-1}$	$3.11 \cdot 10^{-3}$	$2.60 \cdot 10^{-3}$	1.09	1.55	$-1.19 \cdot 10^{-8}$
$5.00 \cdot 10^{-2}$	$9.17 \cdot 10^{-4}$	$7.16 \cdot 10^{-4}$	1.76	1.86	0
$2.50 \cdot 10^{-2}$	$2.42 \cdot 10^{-4}$	$1.81 \cdot 10^{-4}$	1.92	1.98	0

Table: Errors and convergence rates for $\sigma^2/2 = 0.005$, $\Delta t = (\Delta x)^2$.

Numerical results

Non local MFG with analytical solution - Plots

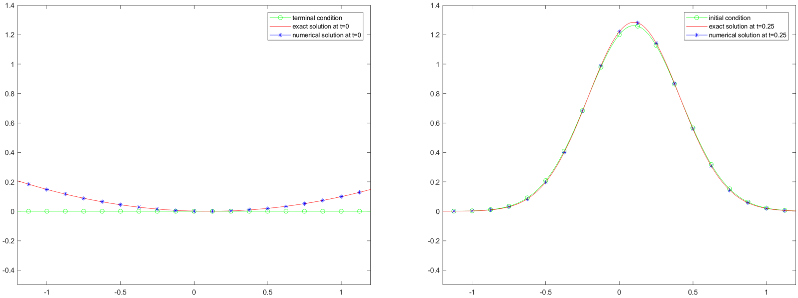

Figure: Solution at time $T = 0$ for the HJ equation (left) and solution at time $T = 0.25$ for the FP equation (right), both with $\sigma^2/2 = 5 \cdot 10^{-3}$, $N = 321$.

Numerical results

Non local MFG with analytical solution - Plots

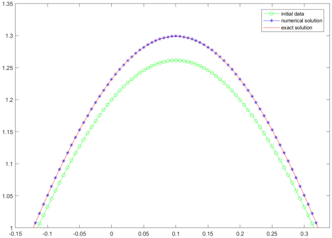

Figure: Detail of the solution at time $T = 0.25$ for the FP equation, with $\sigma^2/2 = 5 \cdot 10^{-3}$, $N = 641$.

Numerical results

Local MFG with reference solution

We consider a smooth problem with the following data:

$$\overline{m}_0(x) = \begin{cases} 4\sin^2(2\pi(x) - \frac{1}{4}) & x \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 0 \quad \text{otherwise}, \end{cases}$$

$$u_T(x) = 0,$$

$$F(x, m(t,x)) = 3\overline{m}_0(x) - \min(4, m(t,x)).$$

- the numerical domain is $[0,T] \times \mathcal{O}_\Delta = (0, 0.05) \times (0,1)$;
- the volatility is $\sigma^2/2 = 0.05$.

In order to check for convergence we computed a reference solution, using $\Delta x = 6.67 \cdot 10^{-4}$ and $\Delta t = (\Delta x)^{3/2}/3$.

Numerical results

Local MFG with reference solution - Errors on the HJB

Δx	Relative Errors on the HJB equation			
	E_∞	E_2	p_∞	p_2
$5.00 \cdot 10^{-2}$	$5.38 \cdot 10^{-2}$	$3.80 \cdot 10^{-2}$	-	-
$2.50 \cdot 10^{-2}$	$1.43 \cdot 10^{-2}$	$1.29 \cdot 10^{-2}$	1.91	1.55
$1.25 \cdot 10^{-2}$	$4.25 \cdot 10^{-3}$	$3.24 \cdot 10^{-3}$	1.74	1.99
$6.25 \cdot 10^{-3}$	$8.84 \cdot 10^{-4}$	$7.99 \cdot 10^{-4}$	2.27	2.01
$3.13 \cdot 10^{-3}$	$3.76 \cdot 10^{-4}$	$3.72 \cdot 10^{-4}$	1.23	1.10
$1.56 \cdot 10^{-3}$	$4.99 \cdot 10^{-5}$	$3.60 \cdot 10^{-5}$	2.90	3.37

Table: Errors and convergence rates for the value function.

Numerical results

Local MFG with reference solution - Errors on the FP

Δx	Relative Errors on the FP equation.			
	E_∞	E_2	p_∞	p_2
$5.00 \cdot 10^{-2}$	$9.07 \cdot 10^{-2}$	$4.82 \cdot 10^{-2}$	-	-
$2.50 \cdot 10^{-2}$	$1.81 \cdot 10^{-2}$	$6.79 \cdot 10^{-3}$	2.32	2.82
$1.25 \cdot 10^{-2}$	$4.81 \cdot 10^{-3}$	$1.36 \cdot 10^{-3}$	1.91	2.32
$6.25 \cdot 10^{-3}$	$7.64 \cdot 10^{-4}$	$2.06 \cdot 10^{-4}$	2.65	2.72
$3.13 \cdot 10^{-3}$	$1.82 \cdot 10^{-4}$	$6.96 \cdot 10^{-5}$	2.07	1.55
$1.56 \cdot 10^{-3}$	$6.28 \cdot 10^{-5}$	$1.24 \cdot 10^{-5}$	1.53	2.49

Table: Errors and convergence rates for the density.

Numerical results

Local MFG with reference solution - Plots

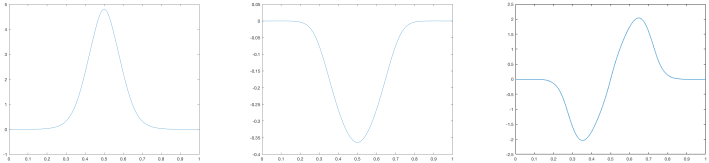

Figure: Density $m(t,x)$ at time $T = 0.05$ (left), the value function $u(0,x)$ (center) and the gradient $u_x(x,0)$ (right) computed with $N = 320$.

Conclusions and future perspectives

Our main aim is to present a new and efficient high-order scheme to solve MFG systems with regular solutions.

- We have developed a new high-order scheme for the (**FP**) equation, based on Lagrange-Galerkin methods combined with a second-order weak approximation of the underlying stochastic characteristic curves.
- We have provided a convergence analysis in the distributional sense and with respect to the weak topology in L^2.
- We have then combined the new scheme for the (**FP**) equation with a high-order semi-Lagrangian scheme for the HJB equation to obtain a high-order scheme for the (**MFG**) system.
- We have shown the performance of the scheme by numerical simulations.

Main advantages: being conservative, explicit, and do not require the standard parabolic CFL condition $\Delta t = O((\Delta x)^2)$ in order to be stable.
Main drawbacks: the loss of positivity for the discrete density and the lack of a constant high-order convergence rate.
Possible future improvements: change basis functions to preserve positivity and to improve the errors.

Thank you for your attention!