Newton schemes for mean field games

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joint work with
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Numerical methods for optimal transport problems,
mean field games, and multi-agent dynamics
January 8-12, 2024

Outline

1. Mean Field Games (MFG)
2. Newton-Semi Lagrangian scheme
3. Newton-Finite Difference scheme
4. Numerical Simulations

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Mean Field Game Problem

Model introduced by Lasry, Lions and Huang, Malhamé, Caines, 2006

$$\begin{cases} -\partial_t u - \frac{\sigma^2}{2}\Delta u + H(x,\nabla u) = F(x,m(t)) & \text{in } [0,T)\times\mathbb{T}^d, \\ \partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\big(\partial_p H(x,\nabla u)m\big) = 0 & \text{in } (0,T]\times\mathbb{T}^d, \\ u(T,\cdot) = u_T(x), \;\; m(\cdot,0) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \qquad \textbf{(MFG)}$$

- $T > 0$, $\sigma \in \mathbb{R}\setminus\{0\}$, $H(x,p)$ is convex and differentiable w.r.t. p
- in the 1st line we have a Hamilton Jacobi Bellman (HJB) equation backward in time
- in the 2nd line we have a Fokker Plank equation forward in time
- m_0 and then $m(t,x)$ represents the density of a probability measure

Mean Field Game Problem

Model introduced by Lasry, Lions and Huang, Malhamé,Caines, 2006

$$\begin{cases} -\partial_t u - \frac{\sigma^2}{2}\Delta u + H(x, \nabla u) = F(m) & \text{in } \mathbb{T}^d \times [0,T), \\ \partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\big(\partial_p H(x, \nabla u) m\big) = 0 & \text{in } \mathbb{T}^d \times (0,T], \\ u(T,\cdot) = u_T(x), \quad m(0,\cdot) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \qquad \textbf{(MFG)}$$

Our aim is

- **to propose a new numerical scheme by discretizing a Newton method in infinite dimension**

Some references of Numerical Approximation of MFG

- Y.Achdou, I.Capuzzo-Dolcetta ('10),Y.Achdou, F.Camilli, I.Capuzzo-Dolcetta ('12), Semi-implicite Finite Difference scheme, Newton Iteration
- E.C.,F.J.Silva ('14, '15) Semi-Lagrangian scheme
- Y.Achdou, M.Lauriere ('20), Combine Continuation Methods with Newton Iterations
- H. Li, Y. Fan, and L. Ying ('21). Multiscale method for mean field games. Second order accurate
- F. Camilli, Q. Tang ('23) Newton's method in infinite dimension for Mean Field Games

Assumptions

We assume $\sigma \neq 0$ and for $\alpha \in (0,1)$

(H1) m_0 is n positive, $m_0 \in \mathcal{P}(\mathbb{T}^d) \cap C^{2+\alpha}(\mathbb{T}^d)$, $u_T \in C^{2+\alpha}(\mathbb{T}^d)$
(H2) F, F', F'' are uniformly bounded mappings from $\mathbb{R}^+ \to \mathbb{R}$.
Moreover, $F'(\cdot) \geq 0$
(H3) $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous, two times differentiable in p and there exists $c, C > 0$ such that

$$cI \leq H_{pp}(x,p) \leq CI, \qquad (x,p) \in \mathbb{T}^d \times \mathbb{R}^d$$

Under **(H1)-(H3)** the MFG system admits one classical solution

Newton method

Following (Camilli Tang 2023) we define the map

$$\mathcal{T} : (u, m) \to \begin{pmatrix} -\partial_t u - \frac{\sigma^2}{2}\Delta u + H(Du) - F(m) \\ \partial_t m - \frac{\sigma^2}{2}\Delta m - \mathrm{div}(m H_p(Du)) \\ u(T) - u_T(x) \\ m(0) - m_0(x) \end{pmatrix}.$$

Then system (**MFG**) is equivalent to

$$\mathcal{T}(u, m) = 0$$

and the corresponding Newton's iterations can be written, at a generic iteration n, as

$$J\mathcal{T}(u^{n-1}, m^{n-1})(u^n - u^{n-1}, m^n - m^{n-1}) = -\mathcal{T}(u^{n-1}, m^{n-1}). \qquad (1)$$

Newton method

The Jacobian of $\mathcal{T}$ is given by

$$J\mathcal{T}(u,m)(v,\rho) = \begin{pmatrix} -\partial_t v - \frac{\sigma^2}{2}\Delta v + H_p(Du)Dv & -F'(m)\rho \\ -\mathrm{div}(mH_{pp}(Du)Dv) & \partial_t\rho - \frac{\sigma^2}{2}\Delta\rho - \mathrm{div}(H_p(Du)\rho) \\ v(T,\cdot) & 0 \\ 0 & \rho(0,\cdot) \end{pmatrix}.$$

Then (1), formally, gets for $n \geq 1$

$$\begin{cases} -\partial_t u^n - \frac{\sigma^2}{2}\Delta u^n + q^n Du^n = q^n Du^{n-1} - H(Du^{n-1}) + F(m^{n-1}) + \\ \qquad F'(m^{n-1})(m^n - m^{n-1}) \\ \partial_t m^n - \frac{\sigma^2}{2}\Delta m^n - \mathrm{div}(m^n q^n) = \mathrm{div}(m^{n-1}H_{pp}(Du^{n-1})(Du^n - Du^{n-1})) \\ m^n(x,0) = m_0(x), \quad u^n(x,T) = u_T(x) \end{cases} \tag{2}$$

where $q^n = H_p(Du^{n-1})$.

Newton method

The Newton method reads:
given (u^0, m^0), find (u^n, m^n) by solving (2) for $n \geq 1$

Theorem (Camilli Tang 2023)

If the initial guess (u^0, m^0) *is close enough to the* (u, m) *solution of* (**MFG**) *, then*

$$\|u - u^n\|_{C^{0,1}} + \|m - m^n\|_{C^0} \leq C^*(\|u - u^{n-1}\|_{C^{0,1}} + \|m - m^{n-1}\|_{C^0})^2. \quad (3)$$

Notation:

$$\|u\|_{C^{0,1}} = \|u\|_{C^0} + \|Du\|_{C^0}$$

For simplicity, next we consider the the Eikonal HJB equation

$$H(x, p) = \frac{p^2}{2} - V(x)$$

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Linearized HJ

Given

$$L(t,x) = \frac{|q^n(t,x)|^2}{2} + F(m^{n-1}(t,x)) + F'(m^{n-1}(t,x))(m^n(t,x) - m^{n-1}(t,x))$$

let us consider

$$\begin{cases} -\partial_t u^n - \frac{\sigma^2}{2}\Delta u^n + q^n D u^n - L(t,x) = 0 \\ u^n(T,x) = u_T(x) \end{cases} \tag{4}$$

Representation formula for $u^n(t,x)$ holds true

$$u^n(t,x) = \mathbb{E}\Big(\int_t^T L(s, X^{t,x}(s)\mathrm{d}s + u_T(X^{t,x}(T))\Big),$$

where

$$X^{t,x}(s) := x - \int_t^s q^n(r, X^{t,x}(r))\mathrm{d}r + \int_t^s \sigma \mathrm{d}W(r) \qquad \text{for all } s \in [t,T].$$

Main Ingredients for SL scheme

- Rapresentation formula in $[t_k, t_{k+1}]$

$$u^n(t_k, x) = \mathbb{E}\left(\int_{t_k}^{t_{k+1}} L(s, X^{t_k,x}(s)\mathrm{d}s\right) + u^n(t_{k+1}, X^{t_k,x}(t_{k+1}))$$

- Semi discretization in time by one-step weak Euler:
  Let us approximate the stochastic characteristics by

$$X^{t_k,x}(t_{k+1}) \simeq x - \Delta t q^n(t_k, x) + \sigma \Delta W,$$

  where $P(\Delta W = \pm\sqrt{\Delta t}) = \frac{1}{2}$

- Trapeizoidal rule for Running cost

$$\int_{t_k}^{t_{k+1}} L(s, X^{t_k,x}(s)\mathrm{d}s \simeq \Delta t L(t_k, x)$$

Ref. Camilli, Falcone(1998), Falcone, Ferretti (2014)

Fully discrete SL scheme for HJL

Let us define $\{u^n_{k,i}\}$ as the solution to

$$\begin{cases} u^n_{k,i} = \frac{1}{2}\sum_{\pm} I[u^n_{k+1}](x_i - \Delta t q^n(t_k, x_i) \pm \sqrt{\Delta t}) + \Delta t L(t_k, x_i) \\ u^n_{N_{\Delta t},i} = u_T(x_i). \end{cases} \quad \text{(SL)}$$

where for a given grid function f

$$I[f](x) := \sum_j \beta_j(x) f_j$$

where $\{\beta_j\}$ denotes the $\mathbb{P}_1$ basis of piecewise linear function, and

$$L(t_k, x_i) = \frac{|q^n(t_k, x_i)|^2}{2} + V(x_i) + F(m^{n-1}_{k,i}) + F'(m^{n-1}_{k,i})(m^n_{k,i} - m^{n-1}_{k,i})$$

where $m^n_{k,i} \simeq m^n(x_i, t_k)$

Backward explicit SL scheme for HJL

For given $Q_k^n = \{q^n(t_k, x_i)\}, M_k^n = \{m_{k,i}^n\}$, compute $U_k^n = \{u_{k,i}^n\}$

$$\begin{cases} U_k^n = \mathcal{A}(Q_k^n)U_{k+1}^n + \Delta t \mathcal{W}_k M_k^n + \Delta t \mathcal{B}_k & k = 0, \dots N_{\Delta t} - 1, \\ U_{N_{\Delta t}}^n = U_T \end{cases} \tag{5}$$

where, given $\{m_{k,i}^{n-1}\}$, we define

$$\begin{cases} (\mathcal{A}(Q))_{ij} := \frac{1}{2} \sum_{\pm} \beta_j(x_i - \Delta t(Q)_i \pm \sqrt{\Delta t}) \\ (\mathcal{W}_k)_{i,j} := F'(m_{k,i}^{n-1})\delta_{i,j} \\ (\mathcal{B}_k)_i := \frac{|q^n(t_k,x_i)|^2}{2} + V(x_i) + F(m_{k,i}^{n-1}) + F'(m_{k,i}^{n-1})m_{k,i}^{n-1} \end{cases}$$

Fully discrete adjoint SL scheme for FP

Given

$$G(t,x) = \mathrm{div}(m^{n-1}(t,x)(Du^n(t,x) - Du^{n-1}(t,x)))$$

let us consider

$$\begin{cases} \partial_t m^n - \frac{\sigma^2}{2}\Delta m^n - \mathrm{div}(m^n q^n) = G(t,x) \\ m^n(x,0) = m_0(x). \end{cases}$$

Using the dualilty property

$$\int L(f) g \mathrm{d}x = \int L^*(g) f \mathrm{d}x$$

of the operators

$$L^*(m) := -\frac{\sigma^2}{2}\Delta m - \mathsf{div}\,(q(x)m)$$

$$L(u) := -\frac{\sigma^2}{2}\Delta u + q(x)^\top Du$$

we derive a scheme for the FP

Adjoint SL scheme for FP

For given $Q_k^n = \{q^n(t_k, x_i)\}, U_k^n = \{u_{k,i}^n\}$, compute $M_k^n = \{m_{k,i}^n\}$

$$\begin{cases} M_{k+1}^n = \mathcal{A}^*(Q_k^n) M_k^n + \Delta t \mathcal{Z}_k U_k^n + \Delta t \mathcal{C}_k, \quad k = 0, \ldots, N_{\Delta t} - 1 \\ M_0^n = M_0, \end{cases}$$

where $\mathcal{A}^*(Q)$ is the transpose of $\mathcal{A}(Q)$ and, for given M_k^{n-1}, $\mathcal{Z}_k$ is the matrix defined such that

$$\begin{aligned} (\mathcal{Z}_k U)_i &= (D_{\Delta x} M_k^{n-1})_i (D_{\Delta x} U_k)_i + (M_k^{n-1})_i (\mathrm{Lap}_{\Delta x} U_k)_i \\ &\simeq \mathrm{div}(m^{n-1}(t_k, x_i)(Du^n(t_k, x_i))) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{C}_k)_i &= -(D_{\Delta x} M_k^{n-1})_i (Q_k^n)_i - (M_k^{n-1})_i (\mathrm{div}_{\Delta x} Q_k^n)_i \\ &\simeq -\mathrm{div}(m^{n-1}(t_k, x_i) q(t_k, x_i)) \end{aligned}$$

Fully discrete Newton scheme

Given (U^{n-1}, M^{n-1}), define $Q_k^n := D_h U_k^{n-1}$ and compute (U^n, M^n) as solution of the linear system

$$\begin{cases} U_k^n = \mathcal{A}(Q_k^n) U_{k+1}^n + \Delta t \mathcal{W}_k M_k^n + \Delta t \mathcal{B}_k & k = 0, \dots N_{\Delta t} - 1, \\ M_{k+1}^n = \mathcal{A}^*(Q_k^n) M_k^n + \Delta t \mathcal{Z}_k U_k^n + \Delta t \mathcal{C}_{k+1}, & k = 0, \dots, N_{\Delta t} - 1 \\ U_{N_{\Delta t}}^n = U_T \quad M_0^n = M_0, \end{cases}$$

(Newton-SL)

which can be written as an Hamiltonian system

$$\begin{pmatrix} \mathbb{A} & -\mathbb{W} \\ -\mathbb{Z} & -\mathbb{A}^\top \end{pmatrix} \begin{pmatrix} \bar{U} \\ \bar{M} \end{pmatrix} = \begin{pmatrix} \mathbb{B} \\ \mathbb{C} \end{pmatrix} \tag{6}$$

Proposition

If $M^n \geq 0$, then for any $n \in \mathbb{N}$ there exists a unique solution (U^n, M^n) to (6)

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Implicit FD scheme for HJL

Given q^n, m^n, m^{n-1}, we define $\{u^n_{k,i}\}$ for $k = 0, \dots N_{\Delta t} - 1$ as the solution to the following Implicit FD scheme

$$\begin{cases} u^n_{k,i} = u^n_{k+1,i} + \Delta t \mu^k_i \mathrm{Lap} u^n_{k,i} + \Delta t q^n(t_k, x_i) D_h u^n_{k,i} + \Delta t L(t_k, x_i) \\ u^n_{N_{\Delta t,i}} = u_T(x_i). \end{cases} \tag{FD}$$

where

$$\mu^k_i = \frac{\sigma^2}{2} + \Delta x |q^n(t_k, x_i)|$$

and

$$L(t_k, x_i) = \frac{|q^n(t_k, x_i)|^2}{2} + V(x_i) + F(m^{n-1}_{k,i}) + F'(m^{n-1}_{k,i})(m^n_{k,i} - m^{n-1}_{k,i})$$

which can be rewritten as

$$\begin{cases} U^n_k = U^n_{k+1} - \Delta t \mathcal{D}_k U^n_k + \Delta t \mathcal{W}_k M^n_k + \Delta t \mathcal{B}_k \qquad k = 0, \dots N_{\Delta t} - 1, \\ U^n_{N_{\Delta t}} = U_T \end{cases} \tag{7}$$

FD Newton scheme

Given (U^{n-1}, M^{n-1}), define $Q_k^n := D_h U_k^{n-1}$ and compute (U^n, M^n) as solution of the linear system, for $k = 0, \ldots, N_{\Delta t} - 1$

$$\begin{cases} U_k^n = U_{k+1}^n - \Delta t(\mathcal{D}_k U_k^n + \mathcal{W}_k M_k^n + \mathcal{B}_k) \\ M_{k+1}^n = M_k^n - \Delta t((\mathcal{D}_k)^* M_{k+1}^n + \mathcal{Z}_{k+1} U_{k+1}^n + \mathcal{C}_{k+1}) \\ U_{N_{\Delta t}}^n = U_T \quad M_0^n = M_0, \end{cases}$$

(Newton-FD)

Proposition

If $M^n \geq 0$, then for any $n \in \mathbb{N}$ there exists a unique solution (U^n, M^n) to (Newton-FD)

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Newton SL Algorithm

Initial guesses M^0, U^0 and tolerances $\delta > 0, \tau > 0$

Compute $Q^0 = D_h u^0$, set $\epsilon_0 = 1$, $n = 0$

While $\epsilon_n > \tau$

Compute M^n, U^n by (Newton-SL)

(Gauss-Seidel with tolerance δ)

Update Q^n, $n = n + 1$

Let $\epsilon_n = ||M_n - M_{n+1}||_\infty + \|U_n - U_{n+1}\|_\infty$

End While

Newton SL Algorithm

Initial guesses M^0, U^0 and tolerances $\delta > 0, \tau > 0$

Compute $Q^0 = D_h u^0$, set $\epsilon_0 = 1$, $n = 0$

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End While

Newton FD Algorithm

Initial guesses M^0, U^0 and tolerances $\delta > 0, \tau > 0$

Compute $Q^0 = D_h u^0$, set $\epsilon_0 = 1$, $n = 0$

While $\epsilon_n > \tau$

Compute M^n, U^n by (Newton-FD)

(Gauss-Seidel with tolerance δ)

Update Q^n, $n = n + 1$

Let $\epsilon_n = ||M_n - M_{n+1}||_\infty + ||U_n - U_{n+1}||_\infty$

End While

Newton FD Algorithm

Initial guesses M^0, U^0 and tolerances $\delta > 0, \tau > 0$

Compute $Q^0 = D_h u^0$, set $\epsilon_0 = 1$, $n = 0$

While $\epsilon_n > \tau$

Compute M^n, U^n by (Newton-FD)

(Gauss-Seidel with tolerance δ)

Update Q^n, $n = n + 1$

Let $\epsilon_n = ||M_n - M_{n+1}||_\infty + \|U_n - U_{n+1}\|_\infty$

End While

MFG 1d

Let us choose $H(p) = \frac{p^2}{2}$, $V(x) = 0$,

$$m_0(x) = \begin{cases} 4\sin^2(2\pi(x - 1/4)) & \text{if} x \in [1/4, 3/4] \\ 0 & \text{otheriwise} \end{cases}$$

$$F(m(x)) = 3m_0(x) - 4\min(4, m), \quad u_T = 0.$$

We set $\sigma = 0.05$, $T = 0.05$, $\tau = \delta = 10^{-4}$ and periodic boundary conditions.

We solve the MFG system using
Newton-SL, Newton-FD, FD-Newton (Ref. Achodou, Capuzzo Dolcetta (2010))
and Fixed Point-SL (Carlini-Silva 2014)

Comparison Newton-SL vs Fixed Point-SL

h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
Newton-SL				
$2.50 \cdot 10^{-2}$	$5.51 \cdot 10^{-2}$	$1.64 \cdot 10^{-1}$	0.61s	6
$1.25 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$	$1.16 \cdot 10^{-1}$	2.77s	7
$6.25 \cdot 10^{-3}$	$1.83 \cdot 10^{-2}$	$6.61 \cdot 10^{-2}$	13.92s	7
$3.125 \cdot 10^{-3}$	$4.50 \cdot 10^{-3}$	$1.41 \cdot 10^{-2}$	80.60s	7
Fixed Point-SL				
$2.50 \cdot 10^{-2}$	$5.75 \cdot 10^{-2}$	$1.62 \cdot 10^{-1}$	8.09s	10
$1.25 \cdot 10^{-2}$	$2.84 \cdot 10^{-2}$	$1.11 \cdot 10^{-1}$	40.79s	10
$6.25 \cdot 10^{-3}$	$2.15 \cdot 10^{-2}$	$5.84 \cdot 10^{-2}$	259.72s	12
$3.125 \cdot 10^{-3}$	$9.50 \cdot 10^{-3}$	$6.51 \cdot 10^{-3}$	2793.71s	12

Table: Errors for the approximation of solution (u, m) using Fixed Point and Newton SL schemes with $\Delta t = h^{3/2}/2$

Newton-FD vs FD-Newton

h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
		FD-Newton		
$2.50 \cdot 10^{-2}$	$1.23 \cdot 10^{-1}$	$3.11 \cdot 10^{-2}$	2.23s	7
$1.25 \cdot 10^{-2}$	$6.21 \cdot 10^{-2}$	$1.63 \cdot 10^{-2}$	18.32s	8
$6.25 \cdot 10^{-3}$	$3.14 \cdot 10^{-2}$	$8.75 \cdot 10^{-3}$	92.91s	8
$3.125 \cdot 10^{-3}$	$1.77 \cdot 10^{-2}$	$9.54 \cdot 10^{-3}$	597.21s	8
		Newton-FD		
$2.50 \cdot 10^{-2}$	$1.532 \cdot 10^{-1}$	$3.42 \cdot 10^{-2}$	1.48s	7
$1.25 \cdot 10^{-2}$	$6.71 \cdot 10^{-2}$	$1.83 \cdot 10^{-2}$	12.27s	7
$6.25 \cdot 10^{-3}$	$3.37 \cdot 10^{-2}$	$9.51 \cdot 10^{-3}$	68.10s	7
$3.125 \cdot 10^{-3}$	$1.91 \cdot 10^{-2}$	$7.38 \cdot 10^{-3}$	436.01s	7

Table: Errors for the approximation of solution (u, m) using FD-Newton and Newton-FD shemes with $\Delta t = h/4$

MFG 1d

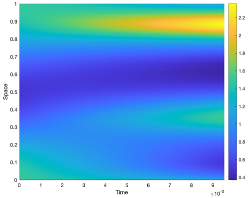
Let us choose $H(p) = p^2$, $V(x) = 200\cos(2\pi x) - 10\cos(4\pi x)$,

$$m_0(x) = 1 + \frac{1}{2}\cos(2\pi x), \quad u_T = \sin(4\pi x) + 0.1\cos(10\pi x),$$

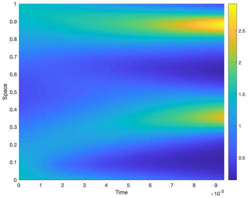
$$F(m(x)) = m^2, \quad .$$

We set $T = 0.01$, $\tau = \delta = 10^{-4}$ and periodic boundary conditions.
We solve the MFG system using Newton-SL, Newton-FD, FD-Newton (Ref. Achodou. Capuzzo Dolcetta 2010) and We set $\Delta t = h$ for the two finite differences schemes, and $\Delta t = h^{3/2}/2$ for Newton-SL

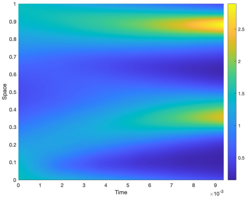
MFG 1d $\sigma = 0.4$



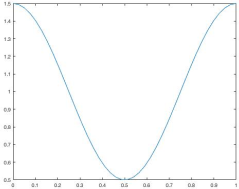
Newton-SL



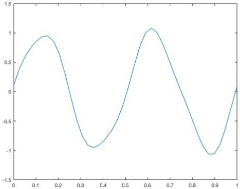
Newton-FD



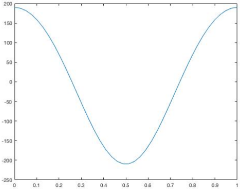
FD-Newton



Initial mass m_0

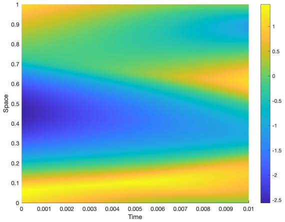


Terminal cost u_T

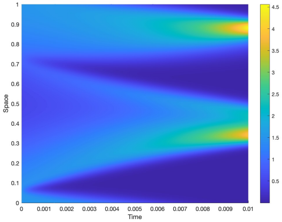


Potential V

MFG, Newton-SL $\sigma = 0.02$



Value Function



Density

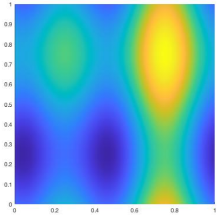
MFG 2d, Newton-SL

Let us choose $H(p) = p^2$, $V(x) = -\sin(2\pi x_1) - \sin(2\pi x_2) - \cos(4\pi x_1)$, $T = 1$, $\tau = \delta = 10^{-4}$, periodic boundary conditions,

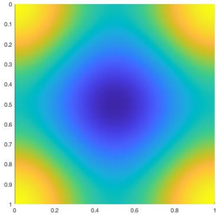
$$m_0(x) = 1 + 12\cos(2\pi x_1) + 12\cos(2\pi x_2), \quad u_T = \cos(2\pi x_1) + \cos(2\pi x_2),$$

$$F(m(x)) = m^2.$$

We solve the MFG system using Newton-SL with $\Delta t = h^2$

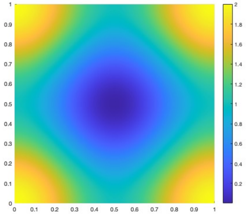


Potential V

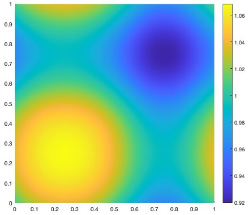


Terminal cost u_T

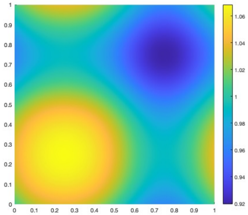
MFG 2d, Newton-SL



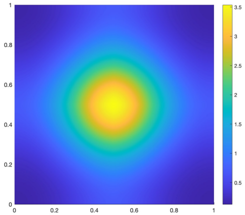
Initial condition m_0



Density at $t = 0.5$



Density at $t = 0.75$



Density at $T = 1$

Conclusions

- FD-Newton and Newton-DF show similar behaviour in terms of CPU time and accuracy
- Newton-SL needs the cheapest CPU time and shows comparable accuracy with respect to the other methods
- Newton-SL scheme works well in hyperbolic regime (σ small)

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