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Hybrid games in route planning for sailing vessels and their mean field limit

Adriano FESTA
adriano.festa@polito.it

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Numerical methods for optimal transport problems, mean field games, and multi-agent dynamics

Universidad Técnica Federico Santa María, Valparaíso, Chile.

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Outline

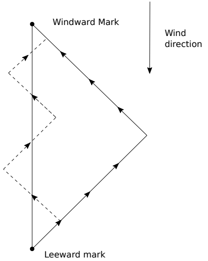
1. **Route planning for sailing vessels**
   - Optimal control of Hybrid differential Games
2. **The match race problem**
3. **From Hybrid control to Hybrid Mean Field Games**
   - A MFG model for optimal sailing for a crowd of vessels

A navigation model: route planning

Basic Goal

The basic objective in a route planning problem is to find the optimal trajectory to move from A to B.

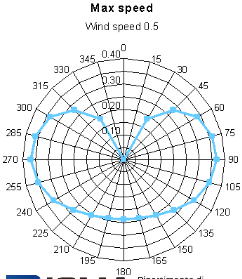
A typical case of interest is when the path A-B is (more or less) *aligned* with the wind direction. In that case the optimal trajectory is not trivially a straight line.

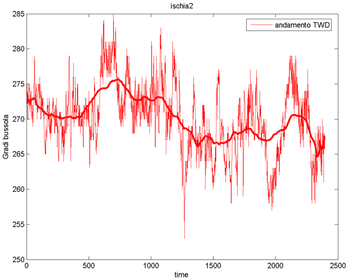


Challenges and motivations

- Sailing boat dynamics - *Complex behaviors*
- Presence of Discontinuous/non convex dynamics
- Presence of noisy data (*Wind*)

Polar plot of the dynamics and wind direction during a race.

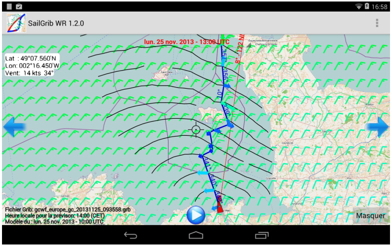


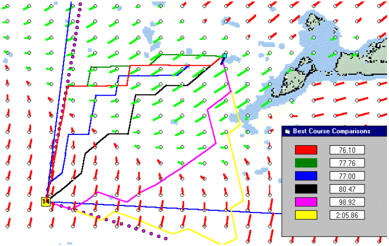


Other challenging aspects

- Change of dynamics - *Change of sails*
- Presence of constraints (islands, etc.)
- Presence of competitors/moving obstacles

Example: *Route planning:* some available software

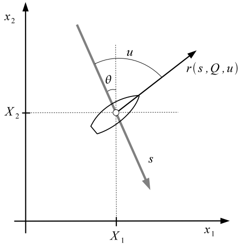


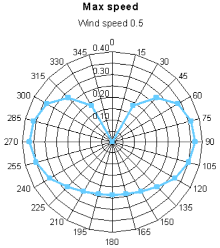


Let be X_1 and X_2 the position of the boat and the control $u \in U = [0, \pi]$ the angle between the boat direction and the wind. The motion of the boat is by

$$\begin{cases} \dot{X}_1(t) = r(s(X,t), Q(t), u(t)) \sin(-\theta(X,t) \pm u(t)) \\ \dot{X}_2(t) = r(s(X,t), Q(t), u(t)) \cos(\theta(X,t) \pm u(t)), \end{cases}$$

where $+$ (starboard tack) and $-$ (port tack).
The function $r : \mathbb{R}_+ \times \mathcal{I} \times [0, \pi] \to \mathbb{R}_+$ models the *polar plot* of the boat.





We assume that the wind direction θ evolves according to

$$d\theta = a(\theta)dt + \bar{\sigma} dW_t.$$

Optimal control of Hybrid differential Games

Let $\mathcal{I} = \{1, 2, \ldots, N_{\mathcal{I}}\}$ and $\mathcal{J} = \{1, 2, \ldots, N_{\mathcal{J}}\}$ be finite, and consider the controlled system X (SDE):

$$\begin{cases} dX(t) = f(X(t), Q(t), a(t), R(t), b(t))dt + \sigma(X(t), Q(t), R(t))\, dW_t, \\ X(0) = x, \;\; Q(0^+) = q, R(0^+) = r, \end{cases}$$

where $x, X \in \mathbb{R}^d$, $q, Q \in \mathcal{I}$, $r, R \in \mathcal{J}$ and dW_t is the differential of a d-dimensional standard Brownian process.

$X(t)$, $Q(t)$ and $R(t)$ are the continuous and the discrete components (one for each player) of the state at time t. The sets of continuous controls are given by:

$$\mathcal{A} = \{a : (0, \infty) \to A \mid a \text{ measurable}, \;\; A \text{ compact}\},$$

$$\mathcal{B} = \{b : (0, \infty) \to B \mid b \text{ measurable}, \;\; B \text{ compact}\}.$$

The function $f : \mathbb{R}^d \times \mathcal{I} \times A \times \mathcal{J} \times B \to \mathbb{R}^d$ is the continuous dynamics.

Switch function

The terms $Q(t)$ and $R(t)$ model the possibility to switch between the various dynamics of the system, that is:

$$\mathcal{Q} = \left\{ Q(\cdot) : (0, \infty) \to \mathcal{I} \mid Q(t) = \sum_i^N q_i \chi_{t_i}(t) \right\},$$

$$\mathcal{R} = \left\{ R(\cdot) : (0, \infty) \to \mathcal{J} \mid R(t) = \sum_i^N r_i \chi_{t_i}(t) \right\},$$

where $\chi_{t_i}(t) = 1$ if $t \in [t_i, t_{i+1})$ and 0 otherwise, $\{t_i\}_{i=1,\dots,N}$ are the (ordered) times at which switches (for either the first or the second player) occur, and $\{q_i\}_{i=1,\dots,N}$ and $\{r_i\}_{i=1,\dots,N}$ are sequences of values in, respectively, $\mathcal{I}$ and $\mathcal{J}$.

The control of the two players are

$$\alpha(t) := (Q(t), a(t)) \in \mathcal{Q} \times \mathcal{A} \text{ and } \beta(t) := (R(t), b(t)) \in \mathcal{R} \times \mathcal{B}.$$

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Hypotheses and strategies

H1

Both f and σ are globally bounded and uniformly Lipschitz continuous w.r.t. x. The discount parameter λ is strictly positive.

Under assumption *H1*, for each $x \in \mathbb{R}^d$, $q \in \mathcal{I}$, $r \in \mathcal{J}$ and two controls $\bar{\alpha} \in \mathcal{Q} \times \mathcal{A}$ and $\bar{\beta} \in \mathcal{R} \times \mathcal{B}$ a solution $X(t)$ of (SDE) is

$$X(t) = x + \int_0^t f(X(s), \bar{Q}(s), \bar{a}(s), \bar{R}(s), \bar{b}(s))ds + \int_0^t \sigma(X(s), \bar{Q}(s), \bar{R}(s))\, dW_s.$$

We use the classic notion of *non-anticipating strategies*:

Definition

A *non-anticipating strategy* for the first (second) player is a map $\phi : \mathcal{R} \times \mathcal{B} \to \mathcal{Q} \times \mathcal{A}$ ($\psi : \mathcal{Q} \times \mathcal{A} \to \mathcal{R} \times \mathcal{B}$) such that, for any $t > 0$, and $\beta(s) = \tilde{\beta}(s)$ ($\alpha(s) = \tilde{\alpha}(s)$) for all $s \leq t$ implies $\phi[\beta](s) = \phi[\tilde{\beta}](s)$ ($\psi[\alpha](s) = \psi[\tilde{\alpha}](s)$) for all $s \leq t$.

The set of the strategies are Φ for the first player and Ψ for the second one.

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Cost functional

The trajectory starts from $(x, q, r) \in \mathbb{R}^d \times \mathcal{I} \times \mathcal{J}$. The two players A and B aim at max/minimizing the cost functional:

$$J(x,q,r;\alpha,\beta) := \mathbb{E}\left(\int_0^{+\infty} e^{-\lambda s}\ell(X(s),Q(s),a(s),R(s),b(s))ds \right.$$
$$\left. + \sum_{i=0}^{N} e^{-\lambda t_i} C\left(Q(t_i^-),Q(t_i^+),R(t_i^-),R(t_i^+)\right)\right).$$

H2
The running cost ℓ is non-negative. Moreover, ℓ and C are bounded and ℓ is uniformly Lipschitz continuous w.r.t. the first argument.

H3
For any choice of $x \in \mathbb{R}^d$ and any $q \in \mathcal{I}$, $r \in \mathcal{J}$, we have that there exists $\bar{c}_0, \hat{c}_0 > 0$ such that $C(q,q,r,r) = 0$,

$$\sup_{q_1 \neq q_2} C(q_1,q_2,r,r) =: -\hat{c}_0 < 0, \qquad \inf_{r_1 \neq r_2} C(q,q,r_1,r_2) =: \bar{c}_0 > 0.$$

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Value function

We are finally ready to define the value function of the game. The *lower value* function $\underline{v}$ of the problem is defined as:

$$\underline{v}(x,q,r) := \inf_{\psi\in\Psi} \sup_{\alpha\in\mathcal{Q}\times\mathcal{A}} J(x,q,r;\alpha,\psi[\alpha]),$$

and the *upper value* $\overline{v}$ is

$$\overline{v}(x,q,r) := \sup_{\phi\in\Phi} \inf_{\beta\in\mathcal{R}\times\mathcal{B}} J(x,q,r;\phi[\beta],\beta).$$

Thanks to a comparison theorem, we know that in general $\underline{v}(x,q,r) \leq \overline{v}(x,q,r)$, while in the case in which $\underline{v}(x,q,r) = \overline{v}(x,q,r)$ we say that *the game has a value* denoted as $v(x,q,r)$.

Differential characterization

DPP provides us some Hamilton–Jacobi–Isaacs equations. Calling

$$H^-(x,q,r,p) := \min_{a\in A}\max_{b\in B}\{-f(x,q,a,r,b)\cdot p - \ell(x,q,a,r,b)\},$$
$$H^+(x,q,r,p) := \max_{b\in B}\min_{a\in A}\{-f(x,q,a,r,b)\cdot p - \ell(x,q,a,r,b)\},$$

and the switching operators $\mathcal{N}$ and $\mathcal{M}$ by:

$$\mathcal{N}[\phi](x,q,r) := \max_{\hat q\neq q}\{\phi(x,\hat q,r) + C(q,\hat q,r,r)\},$$
$$\mathcal{M}[\phi](x,q,r) := \min_{\hat r\neq r}\{\phi(x,q,\hat r) + C(q,q,r,\hat r)\},$$

we have two Isaacs' equations on $\mathbb{R}^d\times\mathcal{I}\times\mathcal{J}$:

$$\max\Big(v(x,q,r)-\mathcal{M}[v](x,q,r), \min\Big(v(x,q,r)-\mathcal{N}[v](x,q,r), \lambda v(x,q,r)+H^-(x,q,r,Dv)-\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^t D^2 v(x,q,r)\right)\Big)\Big)=0, \quad \text{(HJI1)}$$

$$\min\Big(v(x,q,r)-\mathcal{N}[v](x,q,r), \max\Big(v(x,q,r)-\mathcal{M}[v](x,q,r), \lambda v(x,q,r)+H^+(x,q,r,Dv)-\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^t D^2 v(x,q,r)\right)\Big)\Big)=0. \quad \text{(HJI2)}$$

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Definition (Viscosity solutions)

A bounded, uniformly continuous function $v : \mathbb{R}^d \times \mathcal{I} \times \mathcal{J}$ for any choice of $(q, r) \in \mathcal{I} \times \mathcal{J}$, is a *viscosity sub(super)-solution* of the equation (HJI1) if for any $(q, r) \in \mathcal{I} \times \mathcal{J}$ and test function $\gamma \in C^2(\mathbb{R})$ such that $x_0 \in \mathbb{R}^d$ is a local maximum (minimum) point for $v(x, q, r) - \gamma(x)$, then

$$\max\left(v - \mathcal{M}[v], \min\left(v - \mathcal{N}[v], \lambda v + H^-(x, q, r, D\gamma) - \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^t D^2\gamma\right)\right)\right) \leq (\geq)\, 0.$$

Lemma (Costraints)

The following estimates hold:

- *Let $\underline{v}$ be the lower value of the game. We have for any $x \in \mathbb{R}^d$*

$$\mathcal{N}[v](x, q, r) \leq \underline{v}(x, q, r) \leq \mathcal{M}[v](x, q, r), \ \textit{for all}\ (q, r) \in \mathcal{I} \times \mathcal{J}.$$

  Similar relations hold for the upper value $\overline{v}(x, q, r)$.

- *Let v be a viscosity sub(super)-solution of (HJI1). Then, respectively,*

$$v(x, q, r) \leq \mathcal{M}[v](x, q, r) \quad (v(x, q, r) \geq \mathcal{N}[v](x, q, r))\ \forall (q, r) \in \mathcal{I} \times \mathcal{J}.$$

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Theorem

The lower value function $\underline{v}(x, q, r)$ is a viscosity solution of (HJI1). Similarly, upper value function $\overline{v}(x, q, r)$ is a viscosity solution of (HJI2).

Sketch of the proof.

The proof follows the idea that

$$\mathcal{N}[\overline{v}](x, q^*, r) \leq \overline{v}(x, q, r) \leq \mathcal{M}[\overline{v}](x, q, r),$$

while when one of the inequality are strict, and dynamic programming inequality without switching is valid in a neighborhood of the point. For example $\mathcal{N}[\overline{v}](x, q, r) < \overline{v}(x, q, r)$, there exists a $\theta > 0$ and a $(\bar{\alpha}, \bar{\psi}[\bar{\alpha}])$ such that

$$\overline{v}(x, q, r) \leq \mathbb{E}\left(\int_0^\theta \ell(X(s), \bar{\alpha}(s), \bar{\psi}[\bar{\alpha}](s))ds + \overline{v}(X(\tau), \bar{Q}(\tau), \bar{R}(\tau))e^{-\lambda\theta}\right).$$

This means that is in a point (x, q, r), we have $\mathcal{N}[\overline{v}](x, q, r) < \overline{v}(x, q, r) < \mathcal{M}[\overline{v}](x, q, r)$, the dynamic programming principle (without switching) holds in a neighborhood of the point. □

Uniqueness and free loops

H4 (No free loop property). Given a sequence of indices (q_i, r_i) s.t. $(q_1, r_1) = (q_{N+1}, r_{N+1})$ and $q_i = q_{i+1}$, $r_i = r_{i+1}$ are not verified at the same time, then the following must hold true:

$$\sum_{i=1}^{N} C(q_i, q_{i+1}, r_i, r_{i+1}) \neq 0.$$

Theorem (Existence of a value)

Assume H1, H2, H3, H4 and

H5 *(Generalized Isaacs' conditions). Assume that* $H^+ = H^-$ *and additionally, assume for any* $(q, r), (q', r') \in \mathcal{I} \times \mathcal{J}$

$$C(q, q', r, r') = C(q, q, r, r') + C(q, q', r', r') = C(q, q', r, r) + C(q', q', r, r').$$

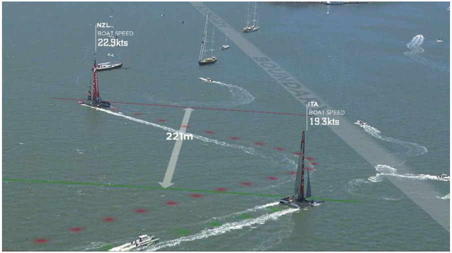
Then, $\overline{v} = \underline{v}$ *is the unique viscosity solution of (HJI1) and (HJI2).*

The Match Race problem

The Match Race is a special form of regatta between two vessels.

Simplified Goal - One leg course

The objective of each player is to maximize its distance from the competitor w.r.t. a vertical direction



Game modeling

To keep the problem formulation in a low dimensional space, we consider the dynamics on their *reduced coordinates*. In addition, we assume the control fixed for both players, so that $a \equiv a^*$ and $b \equiv b^*$ (typically, $a^*, b^* \approx \pi/4$), and the problem reduces to a game with pure switching strategies.

Let $x^A = (x_1^A, x_2^A) \in \mathbb{R}^2$, $x^B = (x_1^B, x_2^B) \in \mathbb{R}^2$ and $\theta \in [-\pi, \pi]$ denote, respectively, the coordinates of the two players and the wind angle. Moreover, denote by $x = x^A - x^B$ the reduced coordinates, and by

$$\phi_q = (-1)^q a^*, \quad \phi_r = (-1)^r b^*$$

the two angles at which the players move w.r.t. the wind, for $q, r \in \mathcal{I} = \mathcal{J} = \{1, 2\}$.

We define the controlled dynamics of the game as before as:

$$\begin{cases} dX^A(t) = f^A(X(t), \Theta(t), Q(t))dt \\ dX^B(t) = f^B(X(t), \Theta(t), R(t))dt \\ d\Theta(t) = \sigma dW(t) \end{cases} \qquad \begin{cases} X^A(0) = x^A \\ X^B(0) = x^B \\ \Theta(0) = \theta \end{cases}$$

for given initial data $x^A, x^B \in \mathbb{R}^2$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, where

$$f^A(x,\theta,q) = s^A(x,\theta+\phi_q)(\cos(\theta+\phi_q), \sin(\theta+\phi_q)),$$
$$f^B(x,\theta,r) = s^B(-x,\theta+\phi_r)(\cos(\theta+\phi_r), \sin(\theta+\phi_r)),$$

where boat speed functions s^A, s^B contain the interaction modeling, e.g.

$$s^P(x,\theta) = \bar{s}^P\left(1 + \min\{s_0^P(x\cdot(\cos(\theta), \sin(\theta))e^{-s_1^P|x|^2}, 0\}\right) \quad (P = A, B),$$

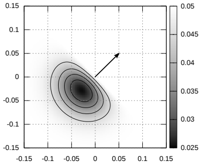


Figure: Speed function: $\bar{s}^P = 0.05$, $s_0^P = 20$ and $s_1^P = 300$, for $\theta = \frac{\pi}{4}$

We call f the deterministic part of the coupled dynamics

$$f(x,\theta,q,r) = \left(f_1^A(x,\theta,q) - f_1^B(x,\theta,r), f_2^A(x,\theta,q) - f_2^B(x,\theta,r), 0 \right).$$

We choose a switching cost C

$$C(q,q',r,r) = -C^A, \qquad C(q,q,r,r') = C^B,$$
$$C(q,q,r,r) = 0, \qquad C(q,q',r,r') = C^B - C^A,$$

for $q,q',r,r' \in \mathcal{I}$, $q \neq q'$, $r \neq r'$, and $C^A, C^B > 0$. and the running cost

$$\ell(x,\theta,q,r) = f_2^A(x,\theta,q) - f_2^B(x,\theta,r),$$

so that the cost functional integrates the vertical component of the relative speed of the two boats.

In this setting (HJI1) and (HJI2) coincide in the form

$$\min \Big(v(q,r) - v(\hat{q},r) + C^A,$$
$$\max \Big(v(q,r) - v(q,\hat{r}) - C^B, \lambda v - f \cdot Dv - \ell - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial \theta^2} v \Big) \Big) = 0 .$$

Tests: Parameters

Parameters for the simulations have been set according to the literature related to **single-hull America's Cup vessels**.

In what follows, the length unit amounts to 1000m, and the time unit to 10s. We choose the bounds $b_1 = 1$, $b_2 = 1$ and $b_3 = \frac{\pi}{4}$, with 201 nodes for each dimension of the grid (i.e., a total number of about $3.2 \cdot 10^7$ nodes).

Concerning the boat speeds, we choose $a^* = b^* = \pi/4$, $\bar{s}^A = \bar{s}^B = 0.05$ and $\bar{s}_1^A = \bar{s}_1^B = 300$. Moreover, we implement starboard/port precedence rule by setting $\bar{s}_0^A = \bar{s}_0^B = 4$ for $q = r$, $\bar{s}_0^A = 4$, $\bar{s}_0^B = 12$ for $q < r$, and $\bar{s}_0^A = 12$, $\bar{s}_0^B = 4$ for $q > r$.

For the **switching costs**, we consider two different settings, a *symmetric* case with $C^A = C^B = 0.02$, and an *asymmetric* case with $C^A = 0.02$ and $C^B = 0.04$.

For the **wind evolution**, we consider a brownian motion with coefficient $\sigma = 0.03$. Finally, we set $\lambda = 0.1$ for the discount factor in the cost functional, and $\Delta t = 0.2$ for the time step in the reconstruction of the optimal trajectories.

Test 1a - Symmetric case

Same initial x_2-coordinate, with the player A (red in the plots) on the left side.

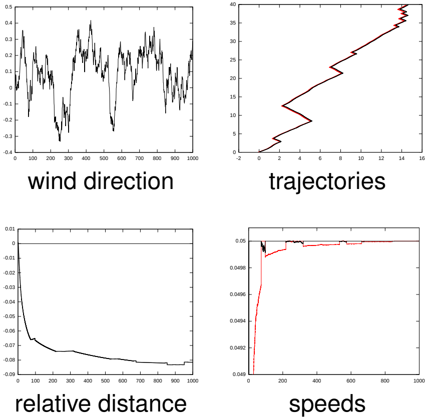


Figure: Test 1a. Symmetric conditions, player B (black trajectory) wins.

Asymmetric switching cost

(race3.mp4)

From Hybrid control to Hybrid Mean Field Games

- Differential game with a continuum number of player
- Main idea: players are indistinguishable and they can optimize their own strategy, knowing the environmental situation, but a single agent can not influence the collective behavior
- The Mean Field Game is obtained by sending to infinite the number of player
- The mean field is given by the collective behavior of the population

From Hybrid control to Hybrid MFG

We consider with the same dynamics as before for a fleet of N boats, i.e.

$$dy^i = f(y^i, q^i, u^i, m^N)dt + \sigma dW_t,$$

where, calling

$$m_1^N := \frac{1}{N_1}\sum_{j=1, q_j=1}^{N} \delta_y^j, \quad m_2^N := \frac{1}{N_2}\sum_{j=1, q_j=2}^{N} \delta_y^j.$$

We note that the function f will depend on the position of all the boats through the definition of m_1^N and m_2^N i.e.

$$s(m^N)(x) = \frac{1}{N_1}\sum_{j=1, q_j=1}^{N} \psi_1(y_j) + \frac{1}{N_2}\sum_{j=1, q_j=2}^{N} \psi_2(y_j).$$

- All the boat similar and therefore "exchangeable".
- Then m_1^N and m_2^N converges in some sense, for $N \to +\infty$ to some $m(x,t,1)$,$m(x,t,2)$.
- Such a space can be endowed with the generalized Wasserstein distance

$$\mathcal{W}(m,m') = \sup\left\{\int_{\mathbb{R}^d} \phi d(m-m') : \phi \in C_c^0(\mathbb{R}^d),\ \|\phi\|_\infty \le 1,\ Lip(\phi) \le 1\right\}.$$

The latter permits us to claim the following:

Theorem

If $\mathcal{E}(|x|) < +\infty$, then, a.s. and in L^1,

$$\lim_{N\to+\infty} \mathcal{W}(m_1^N, m(\cdot,\cdot,1)) = 0, \qquad \lim_{N\to+\infty} \mathcal{W}(m_2^N, m(\cdot,\cdot,2)) = 0.$$

Consequently, the dynamics of the problem converge to

$$dy = f(u,q,m) = \begin{cases} dy_1(t) = r(s(m),u)\sin\left(-\theta + (-1)^{q(t)}u\right)dt \\ dy_2(t) = r(s(m),u)\cos\left(\theta + (-1)^{q(t)}u\right)dt \\ d\theta(t) = a(t)dt + \bar{\sigma}dW_t. \end{cases}$$

which, by Itô's formula, the law m of a solution y, solves in the sense of the distributions the McKean-Vlasov equation

$$\partial_t m(x,t,q) - \frac{1}{2}\sigma\Delta(m(x,t,q)) + \mathrm{div}(f(u,q,m(x,t,q))m(x,t,q)) = -\chi_{\mathcal{S}_q^t} m(x,t,q) + \chi_{\mathcal{S}_{\hat{q}}^t} m(x,t,\hat{q}),$$

$q = 1,2$ and $\hat{q} = 2,1$.

Here the sets $\mathcal{S}_1^t$, $\mathcal{S}_2^t$ are the switching of the states $q = 1$ and $q = 2$ at time t.
Such regions of $\mathbb{R}^3$ possibly change in time and act as sink/source: the portion of mass in $\mathbb{R}^3 \times \{1\}$ that possibly enters the sink instantaneously disappears to re-appear in $\mathbb{R}^3 \times \{2\}$, and viceversa.

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Let two initial distributions on $\mathbb{R}^2$, $m_0^1(x)$ and $m_0^2(x)$ respectively starting with discrete state $q = 1$ and $q = 2$

$$\begin{cases} \partial_t m(x,t,q) - \frac{1}{2}\sigma\Delta(m(x,t,q)) + \mathrm{div}(H_p(x,q,Dv,m)m(x,t,q)) = -\chi_{\mathcal{S}_q^t} m(x,t,q) + \chi_{\mathcal{S}_{\hat{q}}^t} m(x,t,\hat{q}), \\ \max\left(v - \mathcal{N}v, -v_t + H(x,q,Dv,m) + \frac{1}{2}\,\mathrm{tr}\left(\sigma\sigma^t D^2 v\right)\right) = 0, \\ m(x,0,1) = m_0^1(x), \quad m(x,0,q) = m_0^2(x), \\ v(x,T,1) = v(x,T,2) = v_T(x) \end{cases}$$

- [Displacement speed] Adapting what stated before: we s (wind speed) as, for $t \in [0,T]$,

$$s(m) := s(m)(x) = (\psi_1 * m(\cdot,t,1))(x) + (\psi_2 * m(\cdot,t,2))(x).$$

- [Hamiltonian] The hamiltonian for the navigation problem of a single boat against a crowd of self similar vessels is simply

$$H(x,q,p,m) := \sup_{u\in\mathcal{U}} \{-f(u,q,m)\cdot p - 1\}$$

$$f(u,q,m) = \begin{cases} r(s(m),u)\sin\left(-\theta + (-1)^{q(t)}u\right) \\ r(s(m),u)\cos\left(\theta + (-1)^{q(t)}u\right) \\ a(t) \end{cases}$$

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Boundary conditions

The system is **provided with the boundary condition**

$$\begin{cases} m_q(0,x) = m_0^q(x), \\ v(T,x,q) = v_T^{\alpha}(x). \end{cases}$$

A natural choice of the final condition $v_T^q(x)$ is the distance from a desired area of the domain $\Gamma \subset \mathbb{R}$: i.e. $v_T^q(x)$ is the solution of

$$\begin{cases} |v_x(x)| = 1, & x \in \mathbb{R} \setminus \Gamma, \\ v(x) = 0, & x \in \Gamma. \end{cases}$$

Tests

We discuss a simple applicative scenario, where our model **provides the evolution** of the configurations of the system as well as the control strategies $\mathcal{S}$.

We consider a race where the agents aim to reach a buoy placed in $\Gamma := (0, 1.8, x_3)$, for any value of x_3.

The initial configuration of the system is

$$m_0^q(x) = 1/(0.2\sqrt{4\pi})exp(-(x^2 + (y-0.5)^2)/0.2);\ \text{for } q = 1, 2.$$

and the drift of the wind is $a = 0.2$ (rotation right).
The diffusion parameter σ is set to 0.3.

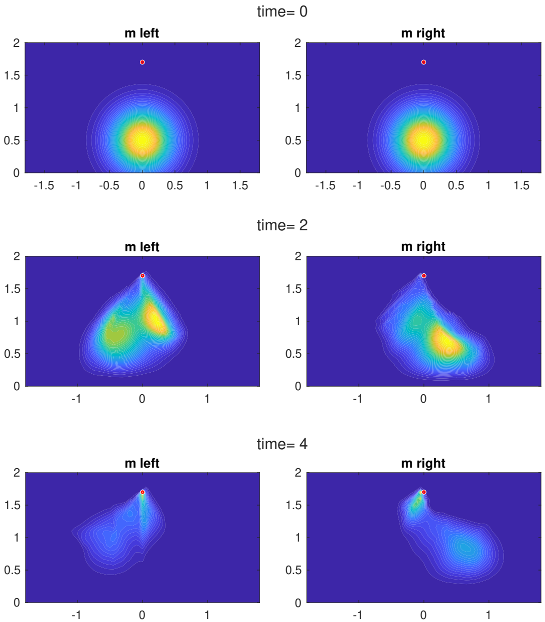

Figure: Density distribution on t=0,2,4.

Other ideas and works

- General framework of hybrid-systems MFG: Well positure (w/wo diffusion). (C. Bertucci, F. Bagagiolo, L. Marzufero, AF)
- Enlarge the framework to **networks** (see traffic & HJ equations, MFG on networks, N. Forcadel, E. Carlini AF)
- Traffic and forecasting devices: a partially informed perspective (P. Goatin AF)

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Some (very partial) bibliograpy

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Thank you for the attention

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