Quantitative mean-field limit for interacting branching diffusions

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Outline

1. Motivation: Lotka-Volterra cross diffusion models
2. Mean-field interacting branching diffusions
3. Main result: convergence rate
4. Ideas of the proof: coupling and optimal transport

1. Motivation: cross diffusion models, local/non-local, Lotka-Volterra

2. Mean-field interacting branching diffusions

3. Main result

4. Idea of the proof

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- PDE model of dispersive spatial interaction between 2 species + competition

$$\begin{cases} \partial_t u - \Delta(d_1 u + a_{11} uu + a_{12} uv) = (r_1 - s_{11} u - s_{12} v) u, \\ \partial_t v - \Delta(d_2 v + a_{21} uv + a_{22} vv) = (r_2 - s_{22} v - s_{21} u) v. \end{cases} \tag{SKT}$$

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- Global (weak) solutions: [Galiano, Garzón, Jüngel 2003], [Chen, Jüngel 2004-2006], [Chen, Daus, Jüngel 2018], [Chen, Jüngel, Wang 2022]. Entropic structure.

- Non-local SKT system introduced in [F', Méléard 2016] :

$$\begin{cases} \partial_t u - \Delta(d_1 u + a_{11}(u * G^{1,1})u + a_{12}(v * G^{1,2})u) = (r_1 - s_{11} u * C^{1,1} - s_{12} v * C^{1,2})u, \\ \partial_t v - \Delta(d_2 v + a_{21}(u * G^{2,1})v + a_{22}(v * G^{2,2})v) = (r_2 - s_{22} v * C^{2,2} - s_{21} u * C^{2,1})v. \end{cases} \tag{NL-SKT}$$

with $G^{i,j}, H^{i,j}, C^{i,j}$ regular kernels.

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Question

Can we quantify the approximation of non-local SKT system by individual based models?

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- Between birth and death events the individual $X^{n,K}$ evolves as the diffusion process

$$\mathrm{d}X_t^{n,K} = b\big(X_t^{n,K}, H * \mu_t^K(X_t^{n,K})\big)\,\mathrm{d}t + \sigma\big(X_t^{n,K}, G * \mu_t^K(X_t^{n,K})\big)\,\mathrm{d}B_t^n.$$

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Theorem (F' , Méléard '16, particular case)

Under Lipschitz regularity on $\sigma, b, G,$ *and* H*, moment assumptions on* $(\mu_0^K)_K$ *and weak convergence* $\mu_0^K \to \mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$*, processes* $(\mu_t^K)_{t\geq 0}$ *converge weakly as* $K \to \infty$ *to* $(\mu_t)_{t\geq 0}$*, the unique weak solution of*

$$\partial_t \mu_t = L^*_{\mu_t}\mu_t + (r - c\langle \mu_t, 1\rangle)\mu_t,$$

with i.c. μ_0*, where*

$$L_\mu f(x) = \frac{1}{2}\mathrm{Tr}\left(a(x, G * \mu(x))\mathrm{Hess}(f)(x)\right) + b(x, H * \mu(x)) \cdot \nabla f(x).$$

Remarks:

- $(N_t^K/K)_{t\geq 0}$ converges to $n_t := \langle \mu_t, 1\rangle$ solution of **logistic** ODE $\dot{n}_t = n_t(r - cn_t)$
- $\bar{\mu}_t := \mu_t/\langle \mu_t, 1\rangle$ satisfies **nonlinear diffusion equation of McKean-Vlasov type**:

$$\partial_t \bar{\mu}_t = L^*_{\mu_t}\bar{\mu}_t$$

  with i.c. $\bar{\mu}_0 := \mu_0/\langle \mu_0, 1\rangle$.

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Assumptions : same as before, plus:

- Conditionally on N_0^K, atoms of μ_0^K are i.i.d. $\sim \bar{\mu}_0 := \mu_0/\langle \mu_0, 1\rangle$, $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$
- $\int_{\mathbb{R}^d} |x|^q \, \mu_0(\mathrm{d}x) < \infty$ for some $q > 2$ and $\sup_K \mathbb{E}(\langle \mu_0^K, 1\rangle^4) < \infty$.
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Theorem (F., Muñoz, 2022, EJP)

For all $K, T > 0$ one has

$$\sup_{t\in[0,T]} \mathbb{E}\left(\left\|\mu_t^K - \mu_t\right\|_{\mathrm{BL}^*}\right) \le C_T \left(I_4(K) + R_{q,d}(K)\right)$$

where $C_T > 0$, $R_{q,d}(K)$ is an explicit polynomial function $\to 0$ as $K \to \infty$ and $I_4(K) = \mathbb{E}\left(|\langle \mu_0^K, 1\rangle - \langle \mu_0, 1\rangle|^4\right)^{\frac{1}{4}}$ is a smaller term.

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$R_{q,d}(K)$ **is explicit...**

Where does the rate $R_{d,q}(K)$ come from?

Recall:

- **Wasserstein distance:** $p \in [1, \infty)$, p-Wasserstein distance $W_p(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ defined by

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \; \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}}.$$

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  $(Z_i)_{i \geq 1}$ i.i.d. $\sim m \in \mathcal{P}(\mathbb{R}^d)$ with $q-$moment $M_q < \infty$, $q > 2$.
  Then, for some $C_{d,q} > 0$ and all $N \in \mathbb{N} \setminus \{0\}$ FIXED,

$$\mathbb{E}\left(W_2^2 \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i}, m \right) \right) \leq C_{d,q} M_q^{\frac{2}{q}} \; R_{d,q}^2(N).$$

Where does the rate $R_{d,q}(K)$ come from?

For all $K > 0$ one has

$$R_{d,q}(K) = \begin{cases} K^{-\frac{1}{4}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d < 4 \text{ and } q \neq 4, \\ K^{-\frac{1}{4}} (\log(1+K))^{\frac{1}{2}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d = 4 \text{ and } q \neq 4, \\ K^{-\frac{1}{d}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d > 4 \text{ and } q \neq \frac{d}{(d-2)}, \end{cases}$$

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- If $\bar{\mu}, \bar{\nu} \in \mathcal{P}(\mathbb{R}^d)$ are normalized versions of $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$, then

$$\begin{aligned} \|\mu - \nu\|_{\mathrm{BL}^*} &\leq \langle \mu, 1 \rangle W_1(\bar{\mu}, \bar{\nu}) + |\langle \mu, 1 \rangle - \langle \nu, 1 \rangle| \\ &\leq \langle \mu, 1 \rangle W_2(\bar{\mu}, \bar{\nu}) + |\langle \mu, 1 \rangle - \langle \nu, 1 \rangle| \end{aligned}$$

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- Bound easily obtained at $t = 0$ from Fournier-Guillin's result and conditional independence of atoms of μ_0^K given mass $\frac{N_0^K}{K}$:

$$\begin{aligned} \mathbb{E}\left(\|\mu_0^K - \mu_0\|_{\mathrm{BL}^*}\right) &\leq C\,\mathbb{E}\Big(\frac{N_0^K}{K} W_2^2(\bar{\mu}_0^K, \bar{\mu}_0)\Big)^{\frac{1}{2}} \mathbb{E}\Big(\frac{N_0^K}{K}\Big)^{\frac{1}{2}} + \mathbb{E}\big(\big|\langle \mu_0^K, 1\rangle - \langle \mu_0, 1\rangle\big|\big), \\ &\leq C\,R_{q,d}(K) + \mathbb{E}\big(\big|\langle \mu_0^K, 1\rangle - \langle \mu_0, 1\rangle\big|\big) \end{aligned}$$

Basic ingredients

- If $\bar{\mu}, \bar{\nu} \in \mathcal{P}(\mathbb{R}^d)$ are normalized versions of $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$, then

$$\begin{aligned} \|\mu - \nu\|_{\mathrm{BL}^*} &\leq \langle \mu, 1 \rangle W_1(\bar{\mu}, \bar{\nu}) + |\langle \mu, 1 \rangle - \langle \nu, 1 \rangle| \\ &\leq \langle \mu, 1 \rangle W_2(\bar{\mu}, \bar{\nu}) + |\langle \mu, 1 \rangle - \langle \nu, 1 \rangle| \end{aligned}$$

- Bound easily obtained at $t = 0$ from Fournier-Guillin's result and conditional independence of atoms of μ_0^K given mass $\frac{N_0^K}{K}$:

$$\begin{aligned} \mathbb{E}\left(\|\mu_0^K - \mu_0\|_{\mathrm{BL}^*}\right) &\leq C\,\mathbb{E}\Big(\frac{N_0^K}{K} W_2^2\left(\bar{\mu}_0^K, \bar{\mu}_0\right)\Big)^{\frac{1}{2}} \mathbb{E}\Big(\frac{N_0^K}{K}\Big)^{\frac{1}{2}} + \mathbb{E}\big(\big|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle\big|\big), \\ &\leq C\, R_{q,d}(K) + \mathbb{E}\big(\big|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle\big|\big) \end{aligned}$$

No longer true for $t > 0$!!!

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- To grant ii) atoms of $\nu_t^K = \frac{1}{K}\sum_{n=1}^{N_t^K} \delta_{Y_t^{n,K}}$ must be independent diffusions

$$\mathrm{d}Y_t^{n,K} = b\big(Y_t^{n,K}, H * \mu_t(Y_t^{n,K})\big)\,\mathrm{d}t + \sigma\big(Y_t^{n,K}, G * \mu_t(Y_t^{n,K})\big)\,\mathrm{d}B_t^n.$$

so that Law$(Y_t^{n,K}) = \bar{\mu}_t$, and we must chose $Y_\tau^{n,K} \sim \bar{\mu}_\tau$ at its birth time τ.

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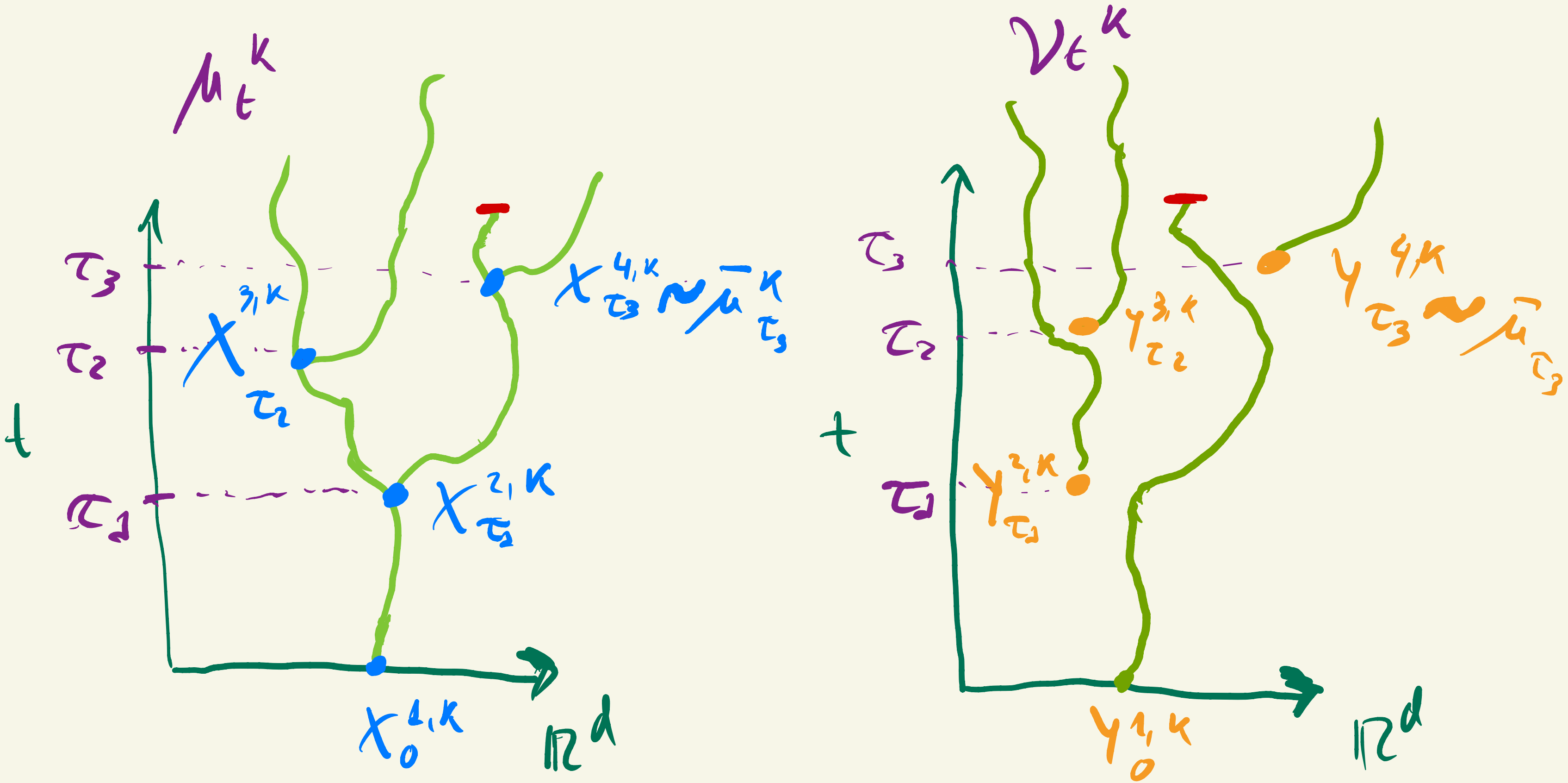
- Drawing
- Triangle ineq. with $\bar{\nu}_{\tau-}^K$ + Gronwall $\Rightarrow$ remainder terms of order $R_{d,q}(K)$ too.



μ_t^K
τ_3
τ_2
τ_1
t
$X^{3,K}_{\tau_2}$
$X^{4,K}_{\tau_3} \sim \bar{\mu}^K_{\tau_3}$
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Thank you!