Existence and numerics for Hughes' model

T. GIRARD

Institut Denis Poisson, Université de Tours

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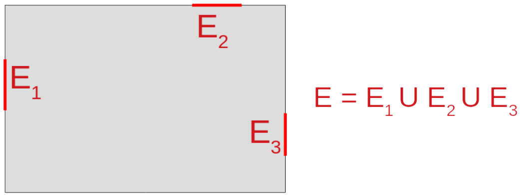
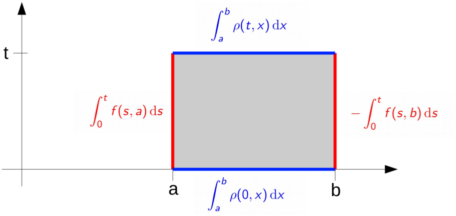
Outline

We want to model a moving crowd. The crowd is represented as a pedestrian density $\rho(t, x)$ between 0 and 1.
Starting at $t = 0$, the pedestrians want to move out of the room the exit(s).



In the one-dimensional case, the agents flux is represented by the flux function f.



$$\int_a^b \rho(t,x)\,\mathrm{d}x = \int_a^b \rho(0,x)\,\mathrm{d}x + \int_0^t f(s,a)\,\mathrm{d}s - \int_0^t f(s,b)\,\mathrm{d}s$$

$$\int_a^b \int_0^t \partial_t \rho(s,x)\,\mathrm{d}s\,\mathrm{d}x = -\int_0^t \int_a^b \partial_x f(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

We end up with:

$$\int_a^b \int_0^t \partial_t \rho(s,x) + \partial_x f(\rho(s,x))\,\mathrm{d}x\,\mathrm{d}s = 0$$

Short version, a scalar conservation law:

$$\rho_t + f(\rho)_x = 0.$$

The flux is equal to the density multiply by the speed of agents.

$$f(s,x) := \rho(s,x)v(s,x)$$

The velocity v is itself governed by the local density:

$$v(s,x) := v_{\max}(1-\rho)$$

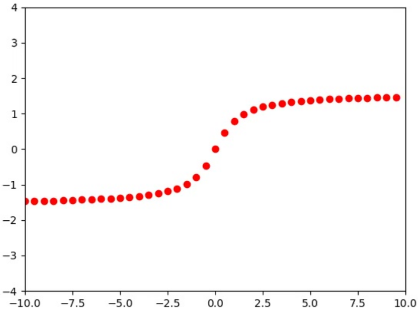
We set $v_{\max} = 1$ and recover:

$$f(s,x) := f(\rho(s,x)) := \rho(s,x)(1-\rho(s,x))$$

• M. J. Lighthill and G. B. Whitham, On kinematic waves. ii. a theory of traffic flow on long crowded roads, (1955).

- **Non-existence of continuous solutions**

We use a method of characteristics to propagate the initial datum:



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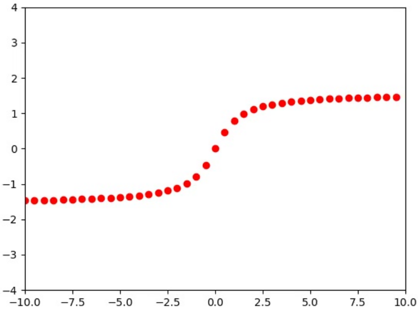


So we consider weak solutions :

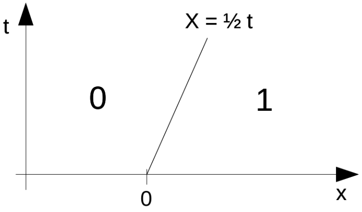
$$\forall \phi \in \mathcal{C}_c^\infty, \quad \iint_{(0,T)\times\mathbb{R}} \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

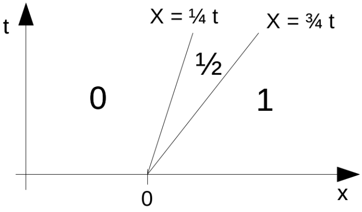
- **Non-uniqueness of weak solutions**

Consider

$$\begin{cases} \rho_t + \left[\rho^2/2\right]_x = 0 \\ \rho(0,x) = \mathbb{1}_{(0,+\infty)} \end{cases} \tag{1}$$

Then the two density functions ρ described below are weak solutions:





Krushkov : entropy conditions

We say that $\rho \in L^\infty$ is an entropy solution to

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ \rho(0,\cdot) = \rho_0(\cdot) \in L^\infty \end{cases}$$

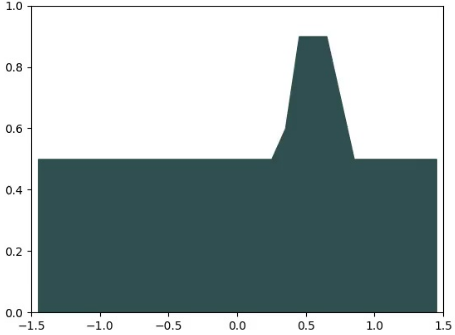
if

$$|\rho - k|_t + (\mathrm{sign}(\rho - k)\,(f(\rho) - f(k)))_x \leq 0 \text{ in the distributional sense.}$$

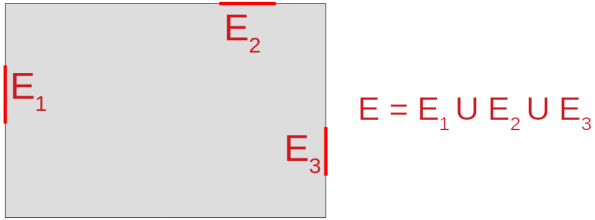
So $\forall k \in \mathbb{R}$, $\forall \phi \in \mathcal{C}_c^\infty$

$$\iint_{(0,T)\times\mathbb{R}} |\rho - k|\phi_t + \mathrm{sign}(\rho - k)\,(f(\rho) - f(k))\,\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0 - k|\phi(0,x)\,\mathrm{d}x \geq 0$$

Interpretation of Kruskov entropy condition in the context of traffic: The admissible shocks correspond to the traffic jams.



Back to the initial problem, at $t = 0$, the agents want to exit the room minimizing their exit time (or total cost...).



Suppose $V(t,x) \in \mathcal{S}^1$ is a vector field corresponding to the choice of direction of an agent located in x at time t. Then the density equation follows from LWR:

$$\rho_t + \mathrm{div}_x(V(t,x)\rho v(\rho)) = 0.$$

How do we compute V ?

For a fixed density $\rho(x)$, we use an optimal control problem.
Fix a density ρ in a given domain Ω. Let $\alpha(\cdot) \in \mathcal{C}^1([0,+\infty), \mathcal{S}^1)$.
Consider the following dynamic for the controlled trajectories y_x solution of the Cauchy problem:

$$\begin{cases} \dot{y}_x(t) = v(\rho(y_x(t)))\alpha(t) \\ y_x(0) = x. \end{cases}$$

In order to model the "disconfort" one can experiment by staying in high density regions, we use a running cost function $g(\rho)$ increasing with respect to the density. Also, since each agent seeks to minimize its exit cost, we assume $g > 0$. We define the value function:

$$\phi(x) = \int_0^\infty g(\rho(y_x(t)))\mathbb{1}_\Omega(y_x(t))\,\mathrm{d}t.$$

Heuristically, suppose that the infinum is a minimum reached for an optimal control $y_x^\star(\cdot)$.

The pedestrian at x should follow the direction field $V(x) = \dot{y}_x^\star(0)$.

Then, using the dynamic programming principle, we should have

$$\dot{y}_x^\star(0) = -\frac{\nabla\phi(x)}{||\nabla\phi(x)||}.$$

For a fixed ρ, using the classical Hamilton-Jacobi-Bellman approach, we want to find the gradient of the viscosity solution the following eikonal equation:

$$||\nabla\phi|| = \frac{c(\rho)}{v(\rho)}.$$

Two big criticism of this model :

• For any t, each agent instantaneously knows the density of the crowd in the whole domain.

• The agents do not anticipate the movement the other pedestrian.

To summarize, we should find the solutions of the Hughes model:

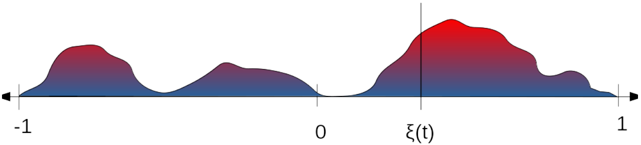
$$\begin{cases} \rho_t + \mathrm{div}_x(\frac{-\nabla\phi}{|\nabla\phi|}\rho v(\rho)) = 0 \\ |\nabla_x \phi| = \frac{g(\rho)}{v(\rho)} \\ \phi(x \in E) = 0 \\ (\nabla_x \phi \cdot n_\Omega)^+ = 0 \text{ if } x \in \partial\Omega \backslash E \\ \rho(0, x) = \rho(x) \end{cases} \tag{2}$$

where n_Ω is the normal unit vector to the boundary of the domain Ω and g is a given cost function depending on the local density.

In the one-dimensional case, we are interested in a corridor $(-1,1)$ with two exits located at $x=\pm 1$.
Then the problem

$$\begin{cases} |\partial_x \phi| = c(\rho) \\ \phi(x=\pm 1) = 0 \end{cases}$$

can be rewriten as an "equilibrium" equation.



We want to solve:

$$\begin{cases} \rho_t + [\mathrm{sign}(x-\xi(t))\rho v(\rho)]_x = 0 \\ \int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x. \end{cases}$$

The curve ξ is called the turning curve.

Theorem

Let $\rho_0 \in L^\infty((-1,1),(0,1))$. *Let* f *verify*

$$f \in W^{1,\infty}((0,1)) \textit{ is concave, non-negative and s. t. } f(0)=0=f(1),$$
$$\operatorname{meas}\Big\{p \in [0,1] \textit{ s.t. } f'(p)=0\Big\} = 0.$$

If the cost c *is affine,*

$$c(\rho) = 1 + \alpha\rho, \quad \alpha > 0,$$

then there exists (ρ, ξ) *a solution to the Hughes problem where* ρ *is a discontinuous-flux entropy solution.*

Proof: a fixed point argument.
The affine cost assumption is an issue : $c(\rho) = \frac{g(\rho)}{v(\rho)}$.

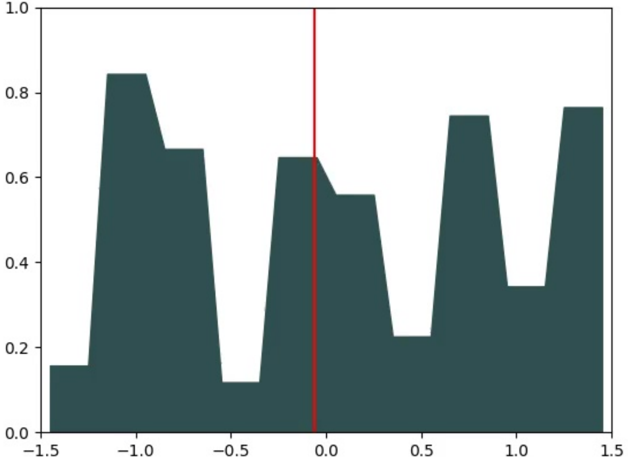
A splitting algorithm :

$\rho_n = \mathcal{FVS}(\xi_n)$ adapted around the turning curve.

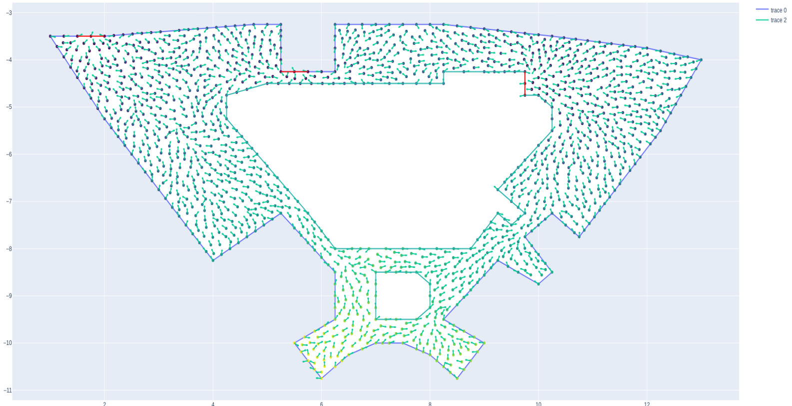
ζ_{n+1} solution to $\displaystyle\int_{-1}^{\zeta_{n+1}} c(\rho_n) = \int_{\zeta_{n+1}}^{1} c(\rho_n)$

$$\xi_{n+1}(s) := \sum_{i=0}^{n} \mathbb{1}_{[i\Delta t,(i+1)\Delta t[}(s) \left(\frac{s - i\Delta t}{\Delta t}\zeta_{i+1} + \frac{(i+1)\Delta t - s}{\Delta t}\zeta_i \right)$$

ξ is one step in time ahead of ρ.


1.0
0.8
0.6
0.4
0.2
0.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

We can approach the eikonal equation's solution via a fast marching numerical scheme. This time we can't easily track the discontinuities so the finite volume scheme is adapted at each edge of the mesh. For fun, here is a simulation for the university restaurant of Tours:



Thank you.