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للعلوم والتقنية
King Abdullah University of
Science and Technology

Price models with common noise

Diogo A. Gomes

with Y. Ashrafyan, J. Gutierrez, M. Lauriére, J. Saude, and R. Ribeiro

www.kaust.edu.sa

Overview

- We study price formation models where agents trade a commodity and interact via its price, ϖ.
- Balance condition is required: supply, Q, equals demand
- Supply may be deterministic or random, e.g., electricity from sustainable sources.
- We can tackle general trading/storage costs

Related references

- Basar and Srikant - revenue maximizing Stackelberg games
- Kizilkale and Malhamé - load adaptive pricing (see also Alasseur, Ben Taher, and Matoussi)
- Gomes and Saúde - deterministic price models
- Cardaliaguet and Lehale - MFG of controls and trade crowding
- Fujii and Takahashi - market clearing conditions with common noise
- Shrivats, Firoozi and Jaimungal - equilibrium pricing in solar renewable energy certificates
- Gomes, Gutierrez, and Ribeiro - quadratic models with common noise
- Ashrafyan, Bakaryan, Gomes, and Gutierrez - potential methods for common noise

Overview

We consider the following price model:

- The model involves numerous agents trading a commodity (such as energy stored in batteries) continuously.
- Agents aim to maximize profit by trading at price $\varpi(t)$, determined by supply-demand balance.
- the supply, $\mathbf{Q}(t)$, is exogenous (and possibly stochastic).

Deterministic Framework

The model involves:

- a price $\varpi \in C([0, T])$
- a value function $u \in C(\mathbb{R} \times [0, T])$
- a path describing the distribution of the agents, $m \in C([0, T], \mathcal{P})$.

NOTE: $\mathcal{P}$ is the set of probabilities on $\mathbb{R}$ with finite second-moment endowed with the 1-Wasserstein distance.

The control problem

- Each agent battery's charge $\mathbf{x}(t)$ changes according to

$$\dot{\mathbf{x}}(t) = \alpha(t).$$

- Each agent selects α to minimize

$$J(x,t,\alpha) = \int_t^T \ell(\mathbf{x}(t), \alpha(s), t)ds + \bar{u}(\mathbf{x}(T)),$$

where ℓ and the terminal cost, $\bar{u}$, are given.

Running cost structure

The Lagrangian takes into account wear and tear and price:

$$\ell(x,\alpha,t) = \ell_0(x,\alpha) + \varpi(t)\alpha(t).$$

For example,

$$\ell_0(x,\alpha,t) = \frac{c}{2}\alpha^2(t) + V(x).$$

Running cost as a price impact

The running term $\frac{c}{2}\alpha^2(t)$ can also be seen as a (temporary) price impact:

- Agents trading at a rate α pay an effective price

$$\varpi + \frac{c}{2}\alpha.$$

Value function

The *value function*, u, is the infimum of J over all bounded measurable controls:

$$u(x,t) = \inf_{\alpha} J(x,t,\alpha).$$

The corresponding *Hamiltonian*, H, is

$$H(x,p) = \sup_{a\in\mathbb{R}} \left(-pa - \ell_0(x,a)\right).$$

The Hamilton–Jacobi equation

From optimal control theory, u is a viscosity solution of

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ u(x, T) = \bar{u}(x). \end{cases}$$

At points of differentiability of u,

$$\alpha^*(t) = -D_p H(\mathbf{x}(t), \varpi(t) + u_x(\mathbf{x}(t), t)).$$

Example

For ℓ_0 as before,

$$H(x,p) = \frac{p^2}{2c} - V(x).$$

So,

$$-u_t + \frac{1}{2c}(\varpi(t) + u_x)^2 - V(x) = 0$$

and the optimal dynamics is

$$\dot{\mathbf{x}} = -\varpi(t) - u_x(\mathbf{x}(t), t).$$

The transport equation

The associated *transport equation* is:

$$\begin{cases} m_t - \left(D_p H(x, u_x + \varpi(t)) m\right)_x = 0, \\ m(x,0) = \bar{m}(x), \end{cases}$$

where $\bar{m}$ is the initial distribution of the agents.
Taking ℓ_0 as before,

$$m_t - \frac{1}{c}(m(\varpi + u_x))_x = 0.$$

Balance condition

We require that demand matches the *energy production function* $\mathbf{Q}(t)$:

$$\int_{\mathbb{R}} \alpha^*(t) m(x,t) dx = \mathbf{Q}(t);$$

that is,

$$\int_{\mathbb{R}} D_p H(x, u_x + \varpi(t)) m(x,t) dx = -\mathbf{Q}(t).$$

This constraint determines the price, $\varpi(t)$.

Deterministic problem

Given $H \in C^\infty$, a supply rate $\mathbf{Q} : [0, T] \to \mathbb{R}$, $\mathbf{Q} \in C^\infty$, solve

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ m_t - (D_p H(x, \varpi(t) + u_x) m)_x = 0 \\ \int_\Omega D_p H(x, \varpi(t) + u_x) dm = -\mathbf{Q}(t), \end{cases}$$

with the initial-terminal conditions

$$\begin{cases} u(x, T) = \bar{u}(x), \\ m(x, 0) = \bar{m}(x), \end{cases}$$

where where $\bar{u}$, $\bar{m}$ are given and $\bar{m}$ is a probability.

Example

For ℓ_0 as before:

$$\begin{cases} -u_t + \frac{1}{2c}(\varpi(t) + u_x)^2 - V(x) = 0 \\ m_t - \frac{1}{c}(m(\varpi + u_x))_x = 0 \\ -\int_{\mathbb{R}} (\varpi + u_x) m(x,t) dx = \mathbf{Q}(t); \end{cases}$$

with the initial-terminal conditions

$$\begin{cases} u(x,T) = \bar{u}(x), \\ m(x,0) = \bar{m}(x). \end{cases}$$

Connection with optimal transport

- Price model/Benamou-Brenier optimal transport (remove red, add blue)

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ m_t - (D_p H(x, \varpi(t) + u_x) m)_x = 0 \\ \int_\Omega D_p H(x, \varpi(t) + u_x) dm = -\mathbf{Q}(t), \end{cases}$$

- Price boundary conditions

$$\begin{cases} u(x, T) = \bar{u}(x), \\ m(x, 0) = \bar{m}(x), \end{cases}$$

- Optimal transport boundary conditions

$$\begin{cases} m(x, T) = \bar{m}_1(x), \\ m(x, 0) = \bar{m}_0(x), \end{cases}$$

Connection with optimal transport

- We can think of the price model as an optimal transport with center of mass constraint
- The price is the Lagrange multiplier for the center of mass constraint

Optimal transport with constraints

The price model equations are the optimality conditions for the minimization problem

$$\min_{(\mu,\nu)\in\mathcal{A}} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \ell(x,\alpha)d\mu(x,\alpha,t) + \int_{\mathbb{R}} \bar{u}d\nu(x),$$

where

$$\begin{aligned}\mathcal{A} = &\Big\{(\mu,\nu)\geq 0 : \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \varphi_t + \alpha\varphi_x(x,t)d\mu \\ &= \int_{\mathbb{R}} \varphi(x,T)d\nu - \int_{\mathbb{R}} \varphi(x,0)d\bar{m}, \quad \forall\varphi\in C^1, \\ &\int_{\mathbb{R}\times\mathbb{R}} \alpha d\mu(x,\alpha) = \mathbf{Q}(t)\Big\}.\end{aligned}$$

The price then becomes a Lagrange multiplier for the demand vs supply balance condition.

Main Result

Theorem (G. and Saúde)

Under natural assumptions, there exists a solution (u, m, ϖ):

- *u is a viscosity solution, Lipschitz and semiconcave in x, and differentiable almost everywhere with respect to m*
- $m \in C([0, T], \mathcal{P})$
- *ϖ is Lipschitz continuous on $[0, T]$.*

Under additional convexity assumptions, the solution is unique.

Finitely many agents

An important case corresponds to finitely many agents

$$m(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(x).$$

Assumptions

Assume the natural convexity conditions

- $(x, \alpha) \mapsto \ell_0(x, \alpha)$ is strictly convex
- $x \mapsto \bar{u}(x)$ is strictly convex

Particle dynamics - N agents

Then, the system

$$\begin{cases} \dot{\mathbf{x}}_i = -D_pH(\mathbf{x}_i, \varpi + \mathbf{p}_i) \\ \dot{\mathbf{p}}_i = D_xH(\mathbf{x}_i, \varpi + \mathbf{p}_i) \\ \frac{1}{N}\sum_{i=1}^{N} D_pH(\mathbf{x}_i, \varpi + \mathbf{p}_i) = -\mathbf{Q}(t) \end{cases}$$

with the boundary conditions

$$\begin{cases} \mathbf{x}_i(0) = x_i \\ \mathbf{p}_i(T) = D_x\bar{u}(\mathbf{x}_i(T)) \end{cases}$$

has a unique solution $(\mathbf{x}, \mathbf{p}, \varpi)$.

Connection with price model

Let (m, u, ϖ) be the solution of the price model with initial-terminal conditions

$$m(x,0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \qquad u(x,T) = \bar{u}(x)$$

Then,

- $\mathbf{p}_i(t) = u_x(\mathbf{x}_i(t), t)$,
- $m(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(x)$.

Constrained minimization

The preceding equations are the Euler-Lagrange equations of the following minimization problem

$$\min \frac{1}{N} \sum_i \left[\int_0^T \ell(\mathbf{x}_i, \dot{\mathbf{x}}_i) ds + \bar{u}(\mathbf{x}_i(T)) \right]$$

under the constraint

$$\frac{1}{N} \sum_i \dot{\mathbf{x}}_i = \mathbf{Q}.$$

The constrained minimization approach gives the existence of a solution by the direct method in the calculus of variations.

Linear-quadratic model - deterministic

Let

$$\ell(t,\alpha) = \frac{c}{2}\alpha^2 + \alpha\varpi(t),$$

where $c > 0$.
The corresponding MFG is

$$\begin{cases} -u_t + \frac{(\varpi(t)+u_x)^2}{2c} = 0 \\ m_t - \frac{1}{c}(m(\varpi(t)+u_x))_x = 0 \\ \frac{1}{c}\int_{\mathbb{R}}(\varpi(t)+u_x)mdx = -\mathbf{Q}(t). \end{cases}$$

Each agent follows optimal trajectories that minimize

$$\int_t^T c\frac{\dot{\mathbf{x}}^2}{2} + \varpi\dot{\mathbf{x}}ds,$$

and, thus, solve the Euler Lagrange equation:

$$c\ddot{\mathbf{x}} + \dot{\varpi} = 0.$$

Integrating, we get the **optimal consumption rule**

$$\dot{\mathbf{x}}(t) = \frac{1}{c}\left(\theta - \varpi(t)\right),$$

for some $\theta \in \mathbb{R}$.

Differentiating the Hamilton-Jacobi equation,

$$-(u_x)_t + (u_x + \varpi)\frac{u_{xx}}{c} = 0,$$

and using the transport equation gives

$$\frac{d}{dt}\int_{\mathbb{R}} u_x m dx = \int_{\mathbb{R}} u_{xt} m + u_x m_t = \int_{\mathbb{R}} u_{xt} m + \frac{1}{c} u_x \left(m(\varpi + u_x)\right)_x$$
$$= \frac{1}{c}\int_{\mathbb{R}} (\varpi + u_x) u_{xx} m - u_{xx} m(\varpi + u_x) dx = 0.$$

Thus, the supply vs demand balance condition becomes

$$\mathbf{Q}(t) = -\frac{1}{c}\int_{\mathbb{R}} (u_x + \varpi) m dx = \frac{1}{c}\left(\Theta - \varpi\right),$$

where

$$\Theta = -\int_{\mathbb{R}} u_x m dx \tag{1}$$

is constant.

Thus, we have the **linear price-supply relation**

$$\varpi = \Theta - c\mathbf{Q}(t). \tag{2}$$

Integrating the optimal consumption rule and taking into account the linear price-supply relation:

$$\mathbf{x}(T) = \mathbf{x}(t) + \frac{1}{c}\int_t^T (\theta - \varpi(s))ds = x + \frac{T-t}{c}(\theta - \Theta) + \int_t^T \mathbf{Q}(s)ds.$$

Accordingly,

$$u(x,t) = \inf_\theta \int_t^T \left[\frac{(\theta - \Theta + c\mathbf{Q}(s))^2}{2c} + \frac{1}{c}(\theta - \Theta + c\mathbf{Q}(s))(\Theta - c\mathbf{Q}(s))\right]$$
$$+ \bar{u}\left(x + \frac{(\theta - \Theta)}{c}(T-t) + K\right),$$

where

$$K = \int_t^T Q(s)ds.$$

By setting $\mu = \theta - \Theta$, for each Θ, we determine u^Θ by solving

$$u^\Theta(x,t) = \inf_\mu \left[\frac{T-t}{2c}\mu^2 + \frac{1}{c}(T-t)\Theta\mu + \int_t^T \left(\Theta - c\frac{\mathbf{Q}(s)}{2} \right) \mathbf{Q}(s)ds \right.$$
$$\left. + \bar{u}\left(x + \frac{\mu}{c}(T-t) + K \right) \right].$$

If $\bar{u}$ is a convex function, there is a unique solution, $\mu(\Theta)$ for each given Θ.

- Optimality conditions in the preceding minimization problem give

$$\mu + \bar{u}_x(\mathbf{x}(T)) = -\Theta, \qquad \mathbf{x}(T) = x + \frac{\mu}{c}(T-t) + K \tag{3}$$

- Given Θ, we solve the Hamilton-Jacobi equation for u^Θ.
- We use the resulting expression for u^Θ in (1) at $t=0$ to get

$$\Theta = -\int_{\mathbb{R}} u_x^\Theta(x,0) m_0(x) dx. \tag{4}$$

- Solving the preceding equation, we obtain Θ and hence ϖ using the price-supply relation.

For example, consider the terminal cost

$$\bar{u}(y) = \frac{\gamma}{2}(y - \zeta)^2 .$$

Solving (3), we obtain

$$\mu = -\frac{\gamma(K + x - \zeta) + \Theta}{1 + \gamma\frac{T-t}{c}}.$$

Accordingly, we have

$$u^{\Theta}(x,t) = \frac{\gamma(K+x-\zeta)^2 + \frac{(t-T)}{c}\Theta(2\gamma(K+x-\zeta)+\Theta)}{2\left(1+\gamma\frac{T-t}{c}\right)} + \Theta K - c\int_t^T \frac{\mathbf{Q}^2(s)}{2}ds.$$

Therefore,

$$u_x(x,t) = \gamma \frac{K + x - \zeta - \frac{(T-t)}{c}\Theta}{1 + \gamma \frac{T-t}{c}}.$$

Using the previous expression for $t = 0$ in (4), we see that Θ solves

$$\Theta = -\gamma \frac{K_0 + \bar{x} - \zeta - \frac{T}{c}\Theta}{1 + \gamma \frac{T}{c}}$$

where

$$\bar{x} = \int_{\mathbb{R}} x m_0 dx$$

and

$$K_0 = \int_0^T Q(s) ds.$$

Thus,

$$\Theta = -\gamma(K_0 + \bar{x} - \zeta).$$

Therefore, using (2), we obtain

$$\varpi = -\gamma(K_0 + \bar{x} - \zeta) - c\mathbf{Q}.$$

Because

$$\dot{\mathbf{x}}(t) = -\frac{\varpi + u_x(\mathbf{x}(t), t)}{c},$$

we have

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t) - \mathbf{x}(t))\gamma}{1 + \frac{T-t}{c}\gamma} + \mathbf{Q} \\ \mathbf{x}(0) = x, \end{cases} \tag{5}$$

where

$$\bar{\mathbf{x}}(t) = \int_{\mathbb{R}} x m(x, t) dx.$$

Averaging (5) with respect to m, we obtain the conservation of energy:

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}(t),$$

Thus, the trajectory of an individual agent is determined by

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t)-\mathbf{x}(t))\gamma}{1+\frac{T-t}{c}\gamma} + \mathbf{Q}(t) \\ \dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}. \end{cases}$$

The previous system is a closed system of ODEs that only involves $\mathbf{Q}$ and the parameters of the problem.

Integrating the transport equation

- Let (u, m, ϖ) solve the price problem.
- The transport equation can be written as $\operatorname{div}_{(t,x)}(m, -H_p(x, \varpi + u_x)m) = 0$.
- Hence, by Poincaré lemma, there exists $\varphi : \Theta \to \mathbb{R}$ such that

$$m = \varphi_x, \quad H_p(x, \varpi + u_x)m = \varphi_t.$$

The potential

- We introduce a potential function $\varphi : \Theta \to \mathbb{R}$ representing, for each t, the cumulative distribution of $m(t,\cdot)$.
- Accordingly, $\varphi_x = m$ and $-\varphi_t$ is the agents' current or flow.

Perspective function

- Let L be the Legendre transform of H, $L(x,v) = \sup_{p\in\mathbb{R}} [-pv - H(x,p)]$.
- Consider the perspective function of L

$$F(x,j,m) = \begin{cases} L\left(x, \frac{j}{m}\right) m, & m > 0 \\ +\infty, & j \neq 0,\ m = 0 \\ 0, & j = 0,\ m = 0. \end{cases}$$

Deterministic potential problem

Find $\varphi : \Theta \to \mathbb{R}$ minimizing

$$\int_{\Theta} F(x, -\varphi_t, \varphi_x) - u_T'(x)\varphi_t \, \mathrm{d}x\mathrm{d}t,$$

over φ s.t. $\varphi(0,x) = \int_{-\infty}^{x} m_0(y)\mathrm{d}y$, and, for all $t \in [0, T]$, $\varphi_x(t,\cdot) \in \mathcal{P}(\mathbb{R})$ and

$$\int_{\mathbb{R}} \varphi(t,x) - \varphi(0,x)\mathrm{d}x = -\int_0^t Q(s)\mathrm{d}s.$$

Differentiating the Hamilton-Jacobi equation with respect to x and using potential, we have

$$\begin{cases} \left(H\left(x, -D_v L\left(x, -\frac{\varphi_t}{\varphi_x}\right)\right)\right)_x \\ \qquad\qquad + \left(D_v L\left(x, -\frac{\varphi_t}{\varphi_x}\right) + \varpi\right)_t = 0, \\ -\int_{\mathbb{R}} \varphi_t + Q\varphi_x \,\mathrm{d}x = 0, \end{cases}$$

with $\varphi_x(0, \cdot) = m_0(\cdot)$ and

$$-D_v L\left(x, -\frac{\varphi_t(T,x)}{\varphi_x(T,x)}\right) - \varpi(T) = u_T'(x).$$

The previous equations are the Euler-Lagrange equations for the functional

$$\int_{\Theta} F(x, -\varphi_t, \varphi_x) - \varpi\,(\varphi_t + Q\varphi_x) - u_T'\varphi_t \; \mathrm{d}x\mathrm{d}t.$$

ϖ can be seen as a Lagrange multiplier for the constraint

$$\int_{\mathbb{R}} \varphi_t + Q\varphi_x \; \mathrm{d}x = 0.$$

Recovery of the value function

Further, let $\varphi \in C^2(\Theta)$ be a solution. Then, we recover the solution (u, m, ϖ) as follows:

$$m = \varphi_x,$$

and

$$u(t,x) = u_T(x) - \int_t^T H\left(x, -D_v L\left(x, \frac{-\varphi_t(s,x)}{\varphi_x(s,x)}\right)\right) \mathrm{d}s.$$

ϖ is given by solving

$$\left(H\left(x, -D_v L\left(x, -\frac{\varphi_t}{\varphi_x}\right)\right)\right)_x + \left(D_v L\left(x, -\frac{\varphi_t}{\varphi_x}\right) + \varpi\right)_t = 0$$

Stochastic Framework

In the stochastic case, the supply process solves

$$dQ_s = b^S(s, Q_s, \varpi_s)ds + \sigma^S(s, Q_s, \varpi_s)dW_s \quad \text{in } [0, T].$$

where

- the drift $b^S : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and volatility $\sigma^S : [0, T] \times \mathbb{R}^2 \to \mathbb{R}_0^+$ are smooth.
- W_s is a standard one-dimensional Brownian motion
- ϖ_s is the price process (to be determined)

A stochastic PDE system for common noise

Find $m : [0, T] \times \mathbb{R} \to \mathbb{R}$, $u, Z : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$, and $\varpi : [0, T] \times \Omega \to \mathbb{R}$ progressively measurable, satisfying $m \geqslant 0$ and

$$\begin{cases} -\mathrm{d}u + H(x, \varpi + u_x)\mathrm{d}t = Z(t, x)\mathrm{d}W(t), \\ u(T, x) = u_T(x), \\ m_t - \left(H_p(x, \varpi + u_x)m\right)_x = 0, \\ m(0, x) = m_0(x), \\ -\int_{\mathbb{R}} H_p(x, \varpi + u_x)m\mathrm{d}x = Q(t). \end{cases}$$

Main highlights

- The previous system couples a Stochastic partial differential PDE with terminal conditions with a PDE with random coefficients.
- Z is a new unknown needed to ensure progressively measurability for u.

Feedback approach

We would like to determine the drift, $b^P : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, and the volatility, $\sigma^P : [0, T] \times \mathbb{R}^2 \to \mathbb{R}_0^+$, so that the price, ϖ_s, solves

$$d\varpi_s = b^P(s, Q_s, \varpi_s)ds + \sigma^P(s, Q_s, \varpi_s)dW_s \quad \text{in } [0, T]$$

and ensures a market clearing condition if all the agents act optimally.

- The feedback approach may fail, but when it works avoids the use of the master equation.
- In particular, we can solve linear-quadratic models, which are important in applications.
- The key technique is the use of an extended state space.

Problem statement - feedback approach

- Given the supply drift, b^S, and supply volatility, σ^S, and $\bar{q}$
- Find u, $\bar{w} \in \mathbb{R}$, the price at $t = 0$, $b^P : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$, and $\sigma^P : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ solving

$$\begin{cases} -u_t + H(x, w + u_x) = b^S u_q + b^P u_w + \frac{1}{2}(\sigma^S)^2 u_{qq} + \sigma^S \sigma^P u_{qw} + \frac{1}{2}(\sigma^P)^2 u_{ww} \\ dm_t = \left(-\operatorname{div}(m\mathbf{b}) + \left(m \frac{(\sigma^S)^2}{2} \right)_{qq} + (m\sigma^S \sigma^P)_{qw} + \left(m \frac{(\sigma^P)^2}{2} \right)_{ww} \right) dt - \operatorname{div}(m\boldsymbol{\sigma}) dW_t \\ \int_{\mathbb{R}^3} q + D_p H(x, w + u_x(x, q, w, t)) m_t(dx \times dq \times dw) = 0, \end{cases}$$

where $\mathbf{b} = (-D_p H(x, w + u_x), b^S, b^P)$, $\boldsymbol{\sigma} = (0, \sigma^S, \sigma^P)$, the divergence is taken w.r.t. (x, q, w), and terminal-initial conditions

$$\begin{cases} u(x, q, w, T) = \Psi(x, q, w) \\ m_0 = \bar{m} = \bar{m}^x \times \delta_{\bar{q}} \times \delta_{\bar{w}}. \end{cases}$$

- The first equation is the Hamilton-Jacobi equation for the extended control problem
- By standard verification arguments, the optimal trajectories solve

$$\begin{cases} d\mathbf{X}_s = -D_pH(\mathbf{X}_s, \varpi_s + u_x(s, \mathbf{X}_s, Q_s, \varpi_s))ds \\ dQ_s = b^S(s, Q_s, \varpi_s)ds + \sigma^S(s, Q_s, \varpi_s)dW_s \\ d\varpi_s = b^P(s, Q_s, \varpi_s)ds + \sigma^P(s, Q_s, \varpi_s)dW_s. \end{cases}$$

- This equation induces a random flow that transports m_0.

Stochastic transport

m solves the stochastic transport equation if

$$\int_{\mathbb{R}^3} \psi(z,t) m_t(dz) = \int_{\mathbb{R}^3} \psi(z,0) m_0(dz) +$$
$$+ \int_0^t \int_{\mathbb{R}^3} \partial_t \psi(z,s) + D_z \psi(z,s) \cdot \mathbf{b}(s,z)$$
$$+ \int_0^t \int_{\mathbb{R}^3} \frac{1}{2} \operatorname{tr}\left(D^2_{zz}\psi(z,s) : (\boldsymbol{\sigma}(s,z), \boldsymbol{\sigma}(s,z))\right) m_s(dz)ds +$$
$$+ \int_0^t \int_{\mathbb{R}^3} D_z \psi(z,s) \cdot \boldsymbol{\sigma}(s,z) m_s(dz) dW_s$$

for any smooth test function $\psi : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$.

Linear-quadratic model - stochastic case

- Suppose that b^S, b^P, σ^S, and σ^P are linear in q and w,H is quadratic, and the terminal condition ψ is quadratic.
- Then

$$-u_t+H(x,w+u_x)=b^S u_q+b^P u_w+\frac{1}{2}(\sigma^S)^2 u_{qq}+\sigma^S\sigma^P u_{qw}+\frac{1}{2}(\sigma^P)^2 u_{ww}$$

has a linear-quadratic solution; that is a solution which is a second-degree polynomial in (x,q,w).

Differential balance condition

For $H(x,p)=\frac{p^2}{2c}$, the balance condition takes the form

$$Q_t=-\varpi_t-\int u_x(x,Q_t,\varpi_t,t)dm.$$

Then

$$dQ_t=-d\varpi-\int\left(u_{xq}\sigma^S+u_{xw}\sigma^P\right)dmdW_t,$$

Because u is quadratic, $u_{xq} = c_{xq}(t)$ and $u_{xw} = c_{xw}(t)$. Thus

$$b^P = -b^S$$

and

$$\sigma^S = -\sigma^P - c_{xq}(t)\sigma^S - c_{xw}(t)\sigma^P$$

Which shows that the problem is solvable.

Example

Let

$$L(\mathbf{v}) = \frac{1}{2}\mathbf{v}^2$$

and set the terminal cost at time $T = 1$

$$\Psi(x) = (x - \alpha)^2.$$

We take $\bar{m}$ to be a normal standard distribution;
We assume the supply is mean-reverting

$$dQ_t = (1 - Q_t)dt + Q_t dW_t,$$

with initial condition $\bar{q} = 1$.

Price dynamics

Therefore, the dynamics for the price becomes

$$d\varpi_t = -(1 - Q_t)dt - \frac{1+a_2^2}{1+a_2^3} Q_t dW_t,$$

where a_2^2 and a_2^3 solve

$$\begin{aligned} \dot{a}_2^2 &= - a_2^3 + a_2^2(1 + 2a_2^1) \\ \dot{a}_2^3 &= 2a_2^1(1 + a_2^3), \end{aligned}$$

with terminal conditions $a_2^2(1) = 0$ and $a_2^3(1) = 0$

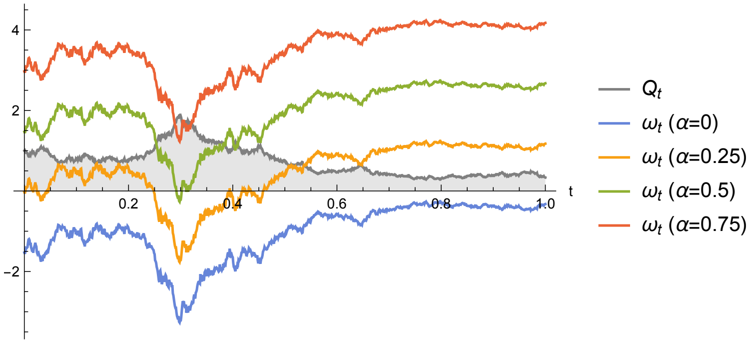


Fig. Supply vs. Price for the values $\alpha = 0$, $\alpha = 0.1$, $\alpha = 0.25$, $\alpha = 0.5$

Finitely many agents

- Let Q, be an L^2 adapted stochastic process with respect to a filtration $\mathbb{F}$
- $\mathbb{H}_{\mathbb{F}}$ is the set of processes $v : [0, T] \times \Omega \to \mathbb{R}$, that are measurable and adapted w.r.t. $\mathbb{F}$, and satisfy $\|v\|^2_{\mathbb{H}_{\mathbb{F}}} < \infty$, where

$$\langle v, w \rangle_{\mathbb{H}_{\mathbb{F}}} := \mathbb{E}\left[\int_0^T v_t w_t dt\right], \quad \|v\|^2_{\mathbb{H}_{\mathbb{F}}} := \langle v, v \rangle_{\mathbb{H}_{\mathbb{F}}}.$$

- Each agent controls trading rate:

$$dX_t = v_t dt, \ \ t \in [0, T],$$

choosing $v \in \mathbb{H}_{\mathbb{F}}$.

Problem formulation

Find a price ϖ and control v^i, all adapted to $\mathbb{F}$, such that for $1 \leqslant i \leqslant N$, X^i solves $dX_t^i = v_t^i dt$, with $X_0^i = x_0^i$, and minimizes the

$$\mathbb{E}\left[\int_0^T L(X_t^i, v_t^i) + \varpi_t v_t^i \; dt + \bar{u}(X_T^i)\right],$$

subject to

$$\frac{1}{N}\sum_{i=1}^N v_t^i = Q_t, \qquad \text{for } 0 \leqslant t \leqslant T.$$

Here ϖ is the Lagrange multiplier for this balance constraint.

- Key tool to prove existence is the direct method in the calculus of variations in $\mathbb{H}_{\mathbb{F}}$.
- Under further convexity conditions uniqueness follows.

Existence of a price

Theorem

Under natural growth and convexity assumptions:

- *There exists a unique minimizer* $v^* \in \mathbb{H}_{\mathbb{F}}^N$
- *consider the corresponding trajectory* X^*. *For* $1 \leqslant i \leqslant N$, *let* $P^i, Z^i \in \mathbb{H}_{\mathbb{F}}$ *solve, on* $[0, T]$,

$$\begin{cases} dP_t^i = -L_x(X_t^{*i}, v_t^{*i})dt + Z_t^i dW_t \\ P_T^i = \bar{u}'(X_T^{*i}). \end{cases}$$

- *There exists a unique* $\Pi \in \mathbb{H}_{\mathbb{F}}$ *such that*

$$\Pi = P^i + L_v(X^{*i}, v^{*i}) \quad \textit{for } 1 \leqslant i \leqslant N.$$

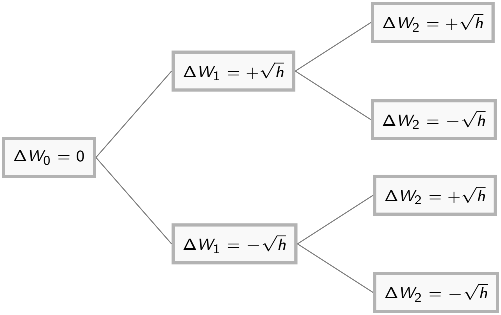
- *Further,* $\varpi = -\Pi$.

Numerical approximations

- Except for quadratic problems, there are no other known solutions.
- Numerical methods are needed and a binomial tree is perhaps one of the easiest ways to do so.

Binomial aproximation

To obtain numerical approximations, we consider a binomial discretization of the driving Brownian motion.



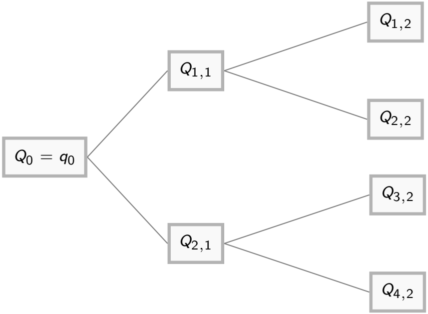
$$\{0, +\sqrt{h}, \; +\sqrt{h}\}$$

$$\{0, +\sqrt{h}, \; -\sqrt{h}\}$$

$$\{0, -\sqrt{h}, \; +\sqrt{h}\}$$

$$\{0, -\sqrt{h}, \; -\sqrt{h}\}$$

Fig. Binomial Tree diagram for $M = 2$ time steps (left) and list of realizations of the noise (right).



$$\{q_0, Q_{1,1}, Q_{1,2}\}$$

$$\{q_0, Q_{1,1}, Q_{2,2}\}$$

$$\{q_0, Q_{2,1}, Q_{3,2}\}$$

$$\{q_0, Q_{2,1}, Q_{4,2}\}$$

Fig. Binomial Tree diagram for $M = 2$ time steps (left) and list of realizations of the supply (right).

- At time t_k, the discrete price process ϖ takes the value ϖ_k, and the measurability condition w.r.t. $\mathcal{F}_k$ means that $\varpi_k \in \left\{\varpi_{1,k}, \ldots, \varpi_{2^k,k}\right\}$, where the values $\varpi_{j,k}$ are unknown
- The controls for each player are also a function of the tree
- At each node, we imposed the balance condition constraint
- We discretize the objective functional in the natural way

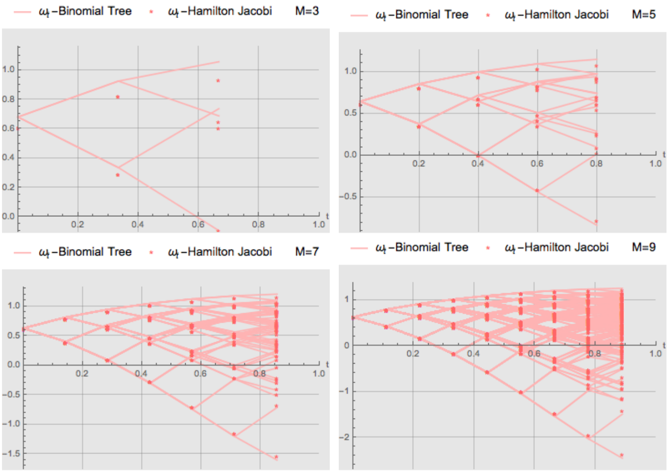


Fig. Binomial Tree and Hamilton-Jacobi approximations for $\eta = 0$ and 3, 5, 7, and 9 time steps.

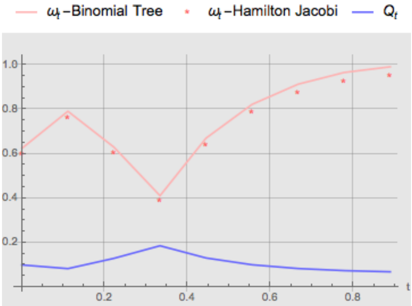


Fig. Sample path of the supply and the corresponding Binomial Tree and Hamilton-Jacobi approximations of the price for $M = 9$ time steps. The L^2 distance between price approximations is $9.16618 * 10^{-2}$.

Motivation for ML approaches

- Stochastic supply price can be approximated numerically by a binomial tree
- Good agreement between numerical results and exact solutions
- However, dimensionality curse limits accuracy.
- Machine learning can improve resolution.

RNN architecture - trading rate

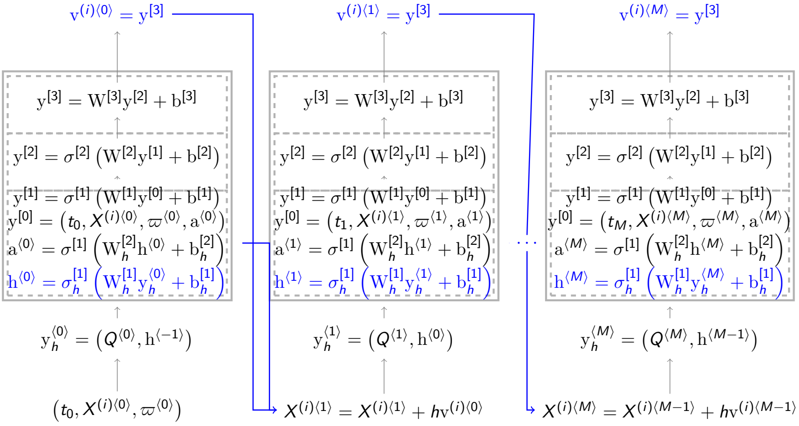


Fig. Iteration of the RNN for v^*, RNN_v, with supply history dependence

RNN price

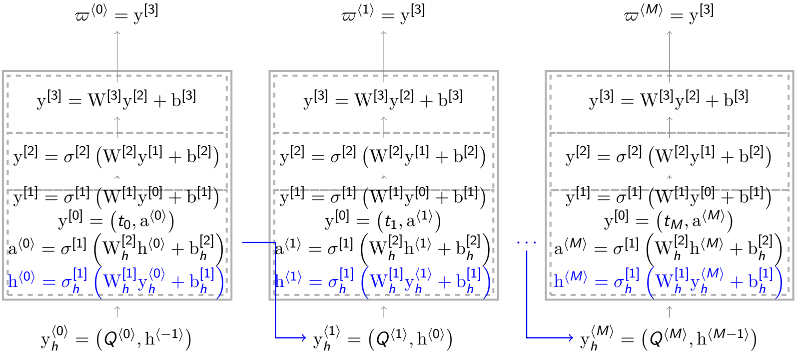


Fig. Iteration of the RNN for ϖ, RNN_ϖ, with supply history dependence

Loss function

We consider the adversarial loss function

$$\mathcal{L}(\Theta_v, \Theta_\varpi) = \frac{1}{N}\sum_{i=1}^{N}\Bigg(\sum_{k=0}^{M-1} h\Big(L(X^{(i)\langle k\rangle}, \mathrm{v}^{(i)\langle k\rangle}(\Theta_v)) + \varpi^{\langle k\rangle}(\Theta_\varpi)\left(\mathrm{v}^{(i)\langle k\rangle}(\Theta_v) - Q^{\langle k\rangle}\right)\Big) + u_T(X^{(i)\langle M\rangle})\Bigg).$$

Using $\mathcal{L}$, we train NN_v and NN_ϖ using an adversarial approach.

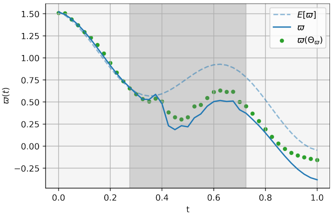
Arrow-Hurwicz-Uzawa like iteration

Key idea:

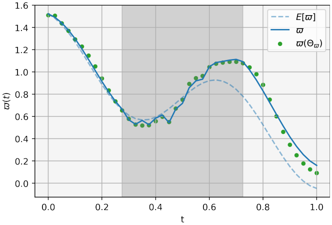
- Perform a descent step in Θ_v
- Perform a ascent step in Θ_{ϖ}.

Common noise RRN training

- To train the RNN, we use a new sample for Q at each SGD step.
- The RNN preserves progressive measurability.



(a) Price realization



(b) Price realization.

Fig. Exact price and RNN approximation. The grey window highlights the times where noise operates.

Common noise - Approximate optimality conditions

The ML framework gives an approximate solution of the optimality conditions

$$\begin{cases} \mathrm{d}\tilde{P}^n(t) = \Big(H_x(\tilde{X}^n(t), \tilde{P}^n(t) + \varpi^N(t)) + \epsilon^n(t)\Big)\,\mathrm{d}t \\ \qquad\qquad +\tilde{Z}^n(t)\mathrm{d}W(t), \\ \tilde{P}^n(T) = u_T'(\tilde{X}^n(T)) - \epsilon_T^n, \\ \mathrm{d}\tilde{X}^n(t) = -H_p(\tilde{X}^n(t), \tilde{P}^n(t) + \tilde{\varpi}^N(t))\mathrm{d}t, \\ \tilde{X}^n(0) = x_0^n, \\ \frac{1}{N}\sum_{n=1}^{N} -H_p(\tilde{X}^n(t), \tilde{P}^n(t) + \tilde{\varpi}^N(t)) = Q(t) + \epsilon_B(t), \end{cases}$$

A posteriori estimates - common noise

Theorem

Let H be uniformly concave-convex in (x, p), separable, with Lipschitz continuous derivatives, u_T is convex with Du_T Lipschitz. Let $(\mathbf{X}, \mathbf{P})$ and ϖ^N solve the N-player price problem with a common noise. Let $(\tilde{\mathbf{X}}, \tilde{\mathbf{P}})$ and $\tilde{\varpi}^N$ be a corresponding approximate solution. Then

$$\|\varpi^N - \tilde{\varpi}^N\| \leqslant C\left(\|\epsilon_H\| + \|\epsilon_B\|\right).$$

Conclusions and future work

- We developed price formation with common noise.
- Our formulation, combined with machine learning techniques, provides a way for solving certain infinite-dimensional MFGs without using the master equation.
- Future work should
  - develop the theory for infinitely many agents with common noise
  - identify better network architectures and convergence results.
  - understand how time-varying preferences affect the model
  - callibration problem

The end

Thanks a lot for your attention! Questions?