From micro to macro: mean field, hydrodynamic and graph limits

SORBONNE UNIVERSITÉ

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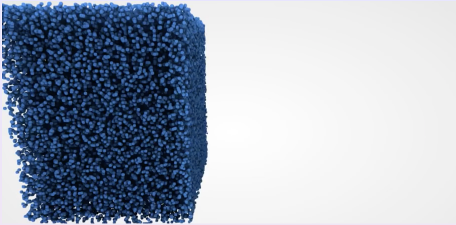
Work with Thierry Paul

Numerical methods for optimal transport problems, mean field games, and multi-agent dynamics, Valparaiso, Jan. 2024

Objective



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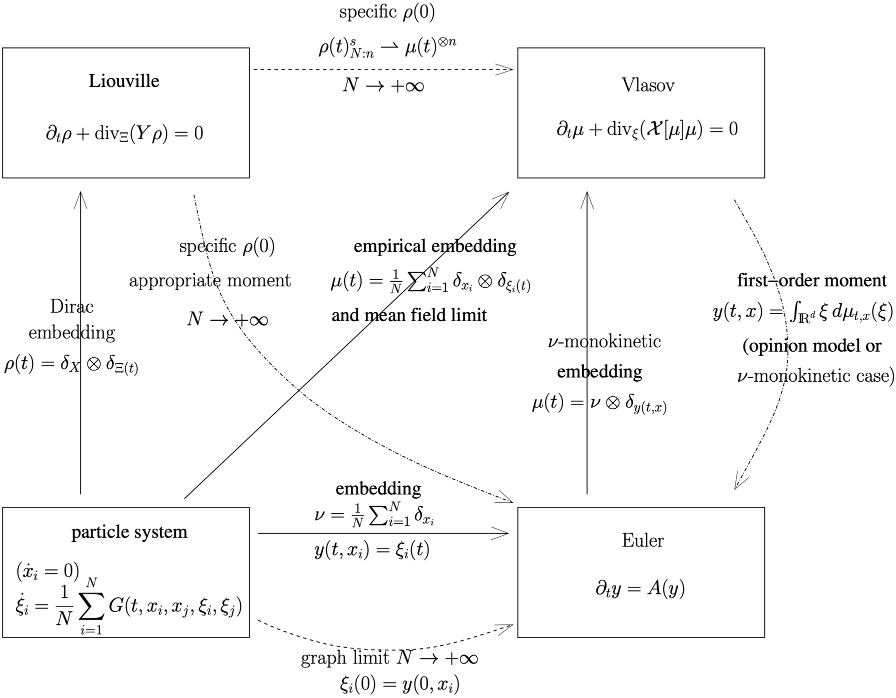
Understand the relationships between finite particle systems with N agents (usually called microscopic scale models) and their various limits as $N \to +\infty$:

- kinetic / mean field limit: Vlasov equation (usually called mesoscopic scale)
- probabilistic lift: Liouville equation
- hydrodynamic / graph limit: Euler equation (usually called macroscopic scale)

As a surprising consequence: *any* (quasi)linear PDE can be obtained as the graph limit of a family of finite particle systems.



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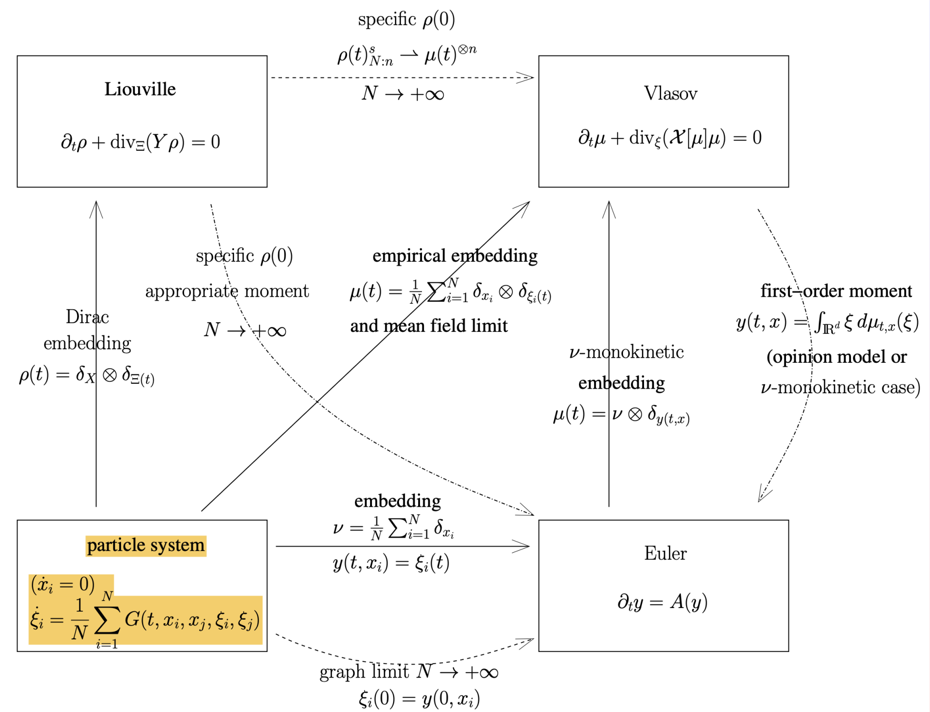
$N \to +\infty$
Liouville
$\partial_t \rho + \mathrm{div}_\Xi(Y\rho) = 0$
Vlasov
$\partial_t \mu + \mathrm{div}_\xi(\mathcal{X}[\mu]\mu) = 0$
specific $\rho(0)$
appropriate moment
$N \to +\infty$
empirical embedding
$\mu(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \otimes \delta_{\xi_i(t)}$
and mean field limit
Dirac
embedding
$\rho(t) = \delta_X \otimes \delta_{\Xi(t)}$
ν-monokinetic
embedding
$\mu(t) = \nu \otimes \delta_{y(t,x)}$
first–order moment
$y(t,x) = \int_{\mathbb{R}^d} \xi \, d\mu_{t,x}(\xi)$
(opinion model or
ν-monokinetic case)
particle system
$(\dot{x}_i = 0)$
$\dot{\xi}_i = \frac{1}{N}\sum_{i=1}^N G(t, x_i, x_j, \xi_i, \xi_j)$
embedding
$\nu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$
$y(t, x_i) = \xi_i(t)$
Euler
$\partial_t y = A(y)$
graph limit $N \to +\infty$
$\xi_i(0) = y(0, x_i)$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Microscopic particle system

$$\boxed{\dot{\xi}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} G_{ij}^N\big(t, \xi_i^N(t), \xi_j^N(t)\big), \qquad i = 1, \ldots, N}$$

$\xi_i^N(t) \in \mathbb{R}^d$

$G_{ij}^N : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$: interaction between the particles i and j

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Examples:

- Hegselmann–Krause first-order consensus (opinion propagation) model:

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} \sigma_{ij}^N \big(\xi_j^N(t) - \xi_i^N(t)\big) \qquad \sigma_{ij}^N \geqslant 0$$

- Cucker–Smale:

$$\dot{q}_i^N(t) = p_i^N(t), \qquad \dot{p}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} a\big(\|q_i^N(t) - q_j^N(t)\|\big)\big(p_j^N(t) - p_i^N(t)\big)$$

- Hamiltonian systems with

$$H^N(q_1, p_1, \ldots, q_N, p_N) = \sum_{j=1}^{N} h_j^N(q_j, p_j) + \frac{1}{N}\sum_{j,k=1}^{N} h_{jk}^N(q_j, p_j, q_k, p_k)$$

Microscopic particle system

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G_{ij}^N(t, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \dots, N$$

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We make the following crucial assumption:

(G) There exist a complete metric space (Ω, d_Ω) and a continuous mapping

$$\begin{array}{rcl} G : \mathbb{R} \times \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R}^d \\ (t, x, x', \xi, \xi') & \mapsto & G(t, x, x', \xi, \xi'), \end{array}$$

loc. Lip. wrt (ξ, ξ') uniformly wrt (t, x, x') on compact sets, and
$\forall N \in \mathbb{N}^* \quad \exists\, x_1, \dots, x_N \in \Omega$ s.t.

$$G(t, x_i, x_j, \xi, \xi') = G_{ij}^N(t, \xi, \xi') \qquad \forall t \in \mathbb{R} \qquad \forall \xi, \xi' \in \mathbb{R}^d \qquad \forall i, j \in \{1, \dots, N\}.$$

(kind of continuous interpolation)

Microscopic particle system

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G_{ij}^N(t, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \ldots, N$$

$\xi_i^N(t) \in \mathbb{R}^d$

$G_{ij}^N : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$: interaction between the particles i and j

Under **(G)**, the particle system is equivalently written as

$$\dot{x}_i^N(t) = 0$$
$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \ldots, N$$

i.e., setting $X^N = (x_1^N, \ldots, x_N^N) \in \Omega^N$ and $\Xi^N(t) = (\xi_1^N(t), \ldots, \xi_N^N(t))$,

$$\dot{\Xi}^N(t) = Y^N(t, X^N, \Xi^N(t))$$

where $Y^N(t, X, \cdot) = (Y_1^N(t, X, \cdot), \ldots, Y_N^N(t, X, \cdot))$ and $Y_i^N(t, X, \Xi) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i, x_j, \xi_i, \xi_j)$

(time-dependent vector field defined on $(\mathbb{R}^d)^N$)

Microscopic particle system

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$\xi_i^N(t) \in \mathbb{R}^d$

$G_{ij}^N : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$: interaction between the particles i and j

Let $(\Phi^N(t, X, \cdot))_{t\in I}$ ($I \subset \mathbb{R}$) be the local-in-time flow of diffeomorphisms of $\mathbb{R}^{dN}$ (particle flow) generated by the time-dependent vector field $Y^N(t, X, \cdot)$.

Lemma (uniform maximal time)

$\forall K \subset \Omega \times \mathbb{R}^d$ compact, $\quad \exists T_{\max}(K) \in (0, +\infty]$ s.t. $\forall N \in \mathbb{N}^*$, $\forall (X, \Xi(0)) \in K^N$, the particle solution $t \mapsto \Phi^N(t, X, \Xi(0))$ is well defined on $[0, T_{\max}(K))$.

Moreover, $\forall T \in [0, T_{\max}(K))$, the set $\Phi^N([0, T] \times K^N)$ is contained in a compact subset of $\mathbb{R}^d$ depending on T but not on N.

Microscopic particle system

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Example: for the Hegselmann Krause opinion propagation model:

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} \sigma_{ij}^N(\xi_j^N(t) - \xi_i^N(t)) = \frac{1}{N}\sum_{j=1}^{N} \sigma(x_i^N, x_j^N)(\xi_j^N(t) - \xi_i^N(t))$$

Assumption **(G)** requires that:

$\exists\Omega$ and $\sigma \in \mathscr{C}^0(\Omega^2)$ s.t. $\forall N \in \mathbb{N}^*$, $\exists\, x_1^N, \ldots, x_N^N \in \Omega$ s.t. $\sigma(x_i^N, x_j^N) = \sigma_{ij}^N (\geqslant 0)$.

We have then $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$, and $T_{\max}(K) = +\infty$.

At the end: graph limit

Two points of view

Particle system: $$\dot\xi_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \dots, N \to +\infty$$

- Riemann sum limit $\xi_i^N(t) \simeq y(t, x_i^N)$ (graph limit)

  one (deterministic) opinion assigned to each agent i

  $$\rightsquigarrow \boxed{\partial_t y(t,x) = A(t, y(t,x)) = \int_\Omega G(t, x, x', y(t,x), y(t,x'))\, d\nu(x')}$$

- Liouville paradigm

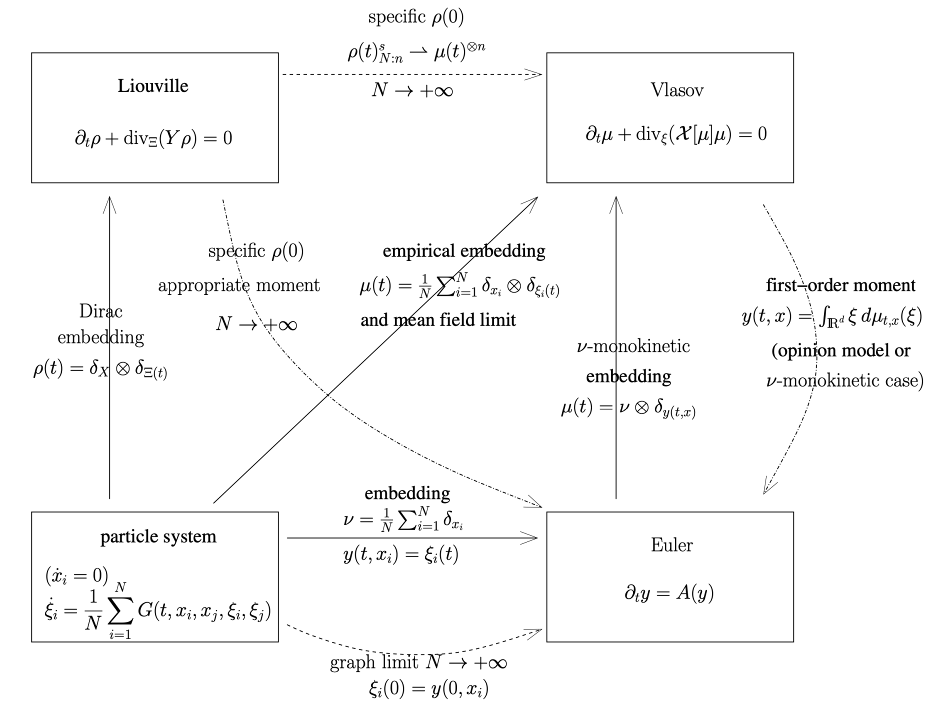
  random opinion assigned to each agent i

  then take marginals ($\rightsquigarrow$ mean field limit)

  then take *hydrodynamic limit* $\rightsquigarrow$ same equation

$$GRAPH\ LIMIT = EULER$$

averaging randomness $\rightsquigarrow$ determinism



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From micro (particle) to macro (Euler): graph limit

Notion of graph limit: introduced by [Medvedev, SIMA 2014] and used recently by:

[Biccari Ko Zuazua, M3AS 2019], [Esposito Patacchini Schlichting Slepcev, ARMA 2021], [Ayi Pouradier Duteil, JDE 2021], [Boudin Salvarani Trélat, SIMA 2022], [Bonnet Pouradier Duteil Sigalotti, M3AS 2022]

Tagged partition associated with $\nu \in \mathcal{P}(\Omega)$

$\forall N \in \mathbb{N}^*$, we say that $(\mathcal{A}^N, X^N)$ is a *tagged partition* of Ω associated with ν if

- $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$ with disjoint subsets $\Omega_i^N \subset \Omega$ s.t.

$$\Omega = \bigcup_{i=1}^{N} \Omega_i^N \qquad \text{with} \qquad \nu(\Omega_i^N) = \frac{1}{N} \quad \text{and} \quad \mathrm{diam}(\Omega_i^N) \leqslant \frac{C_\Omega}{N^r} \qquad \forall i$$

  for some $C_\Omega > 0$ and $r > 0$ not depending on N.
- $X^N = (x_1^N, \dots, x_N^N)$ with $x_i^N \in \Omega_i^N$.

Riemann sum convergence theorem:

$$\forall f \ \nu\text{-Riemann integrable,} \qquad \int_\Omega f \, d\nu = \frac{1}{N} \sum_{i=1}^{N} f(x_i^N) + \mathrm{o}(1)$$

as $N \to +\infty$.

From micro (particle) to macro (Euler): graph limit

The graph limit (i.e., taking the limit of the Riemann sum) of the particle system

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \dots, N$$

as $N \to +\infty$ is the

Euler equation

$$\boxed{\partial_t y(t,x) = A(t, y(t,x)) = \int_\Omega G(t, x, x', y(t,x), y(t,x'))\, d\nu(x')}$$

Proposition

Assume Ω compact. Let $\nu \in \mathcal{P}(\Omega)$ and $y^0 \in L^\infty_\nu(\Omega, \mathbb{R}^d)$. Set $K = \Omega \times \mathrm{ess.im}(y^0)$.
The Euler equation has a unique solution on $[0, T_{\max}(K))$ such that $y(0,\cdot) = y^0(\cdot)$.

(will follow from the next results)

From micro (particle) to macro (Euler): graph limit

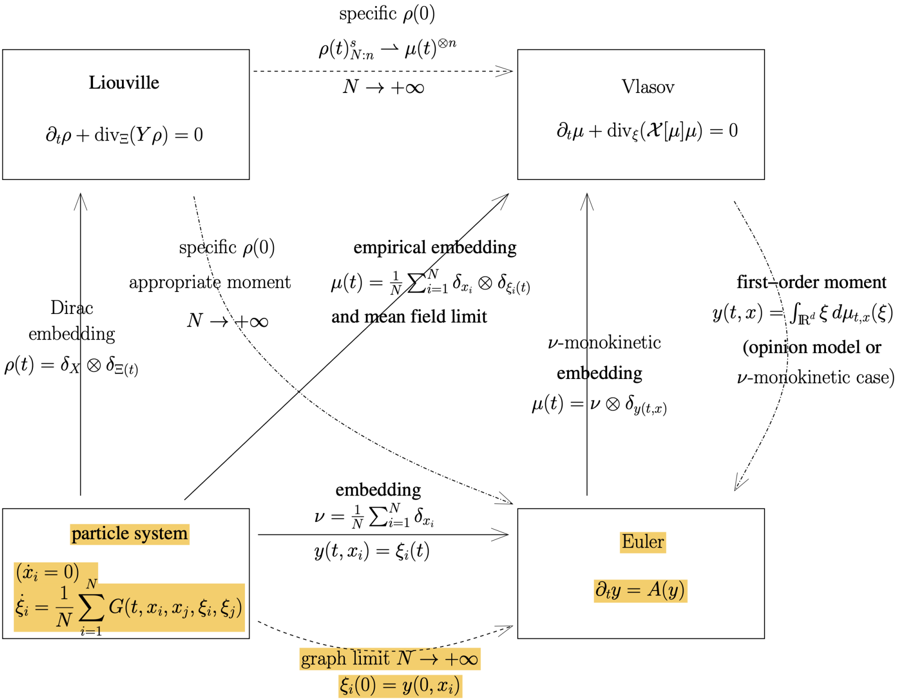
Example: for the Hegselmann–Krause opinion propagation model:

$$\dot{\xi}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma(x_i^N, x_j^N)(\xi_j^N(t) - \xi_i^N(t))$$

we have $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$, and the Euler equation (graph limit) is

$$\partial_t y(t, x) = Ay(t, x) = \int_\Omega \sigma(x, x')(y(t, x') - y(t, x))\, dx'$$

Spectral properties of the bounded operator A studied in [Boudin Salvarani Trélat, SIMA 2022]
$\Rightarrow$ consensus results.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From micro to macro: graph limit

Theorem (start with y^0)

Let $y^0 \in L^\infty_\nu(\Omega, \mathbb{R}^d)$ and let y solution of Euler s.t. $y(0,\cdot) = y^0(\cdot)$.

For any $N \in \mathbb{N}^*$, set

$$y^N(t,x) = \sum_{i=1}^{N} \xi_i^N(t)\, \mathbb{1}_{\Omega_i^N}(x)$$

where $(\xi_1^N(t), \dots, \xi_N^N(t))$ solution of the particle system s.t. $\xi_i^N(0) = y^0(x_i^N) \quad \forall i$.

- If y^0 is ν-Riemann integrable then

$$\left\| y(t,\cdot) - y^N(t,\cdot) \right\|_{L^\infty(\Omega,\mathbb{R}^d)} = \mathrm{o}(1)$$

as $N \to +\infty$, uniformly on compact intervals.

- If G is loc. Lipschitz / (x, x', ξ, ξ') and y^0 is Lipschitz then

$$\left\| y(t,\cdot) - y^N(t,\cdot) \right\|_{L^\infty(\Omega,\mathbb{R}^d)} \leqslant 2\frac{C_\Omega}{N^r}\left(1 + \mathrm{Lip}(y^0)\right) e^{2t\,\mathrm{Lip}(G)}$$

(Lipschitz constant on the supports of y and y^N)

From micro to macro: graph limit

A second result is:

Theorem (start with Ξ_0)

$\forall N \in \mathbb{N}^*$, let $\Xi_0^N \in \mathbb{R}^{dN}$ and let $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$ solution of the particle system s.t. $\Xi^N(0) = \Xi_0^N$. We set as before

$$y^N(t,x) = \sum_{i=1}^{N} \xi_i^N(t)\, \mathbb{1}_{\Omega_i^N}(x).$$

Let y_N solution of Euler s.t. $y_N(0,\cdot) = y^N(0,\cdot)$ (i.e., $y_N(0,x) = \xi_i^N(0)$ if $x \in \Omega_i^N$). Then:

$$\|y^N(t,\cdot) - y_N(t,\cdot)\|_{L^\infty(\Omega,\mathbb{R}^d)} = \mathrm{o}(1)$$

as $N \to +\infty$, uniformly on compact intervals.

If moreover G is loc. Lipschitz / (x, x', ξ, ξ') then

$$\|y^N(t,\cdot) - y_N(t,\cdot)\|_{L^\infty(\Omega,\mathbb{R}^d)} \leqslant 2\frac{C_\Omega}{N^r} e^{2t\,\mathrm{Lip}(G)}$$

(Lipschitz constant on the supports of y_N and y^N)

From micro to macro: graph limit

Sketch of proof of the first theorem in the Lipschitz case:

By definition, $\partial_t y(t,z) = \int_\Omega G(t,z,x'',y(t,z),y(t,x''))\,d\nu(x'')$, hence

$$\partial_t y(t,x) - \partial_t y(t,x') = \int_\Omega G(t,x,x'',y(t,x),y(t,x''))\,d\nu(x'') - \int_\Omega G(t,x',x'',y(t,x),y(t,x''))\,d\nu(x'')$$
$$+ \int_\Omega G(t,x',x'',y(t,x),y(t,x''))\,d\nu(x'') - \int_\Omega G(t,x',x'',y(t,x'),y(t,x''))\,d\nu(x'')$$

hence

$$\|\partial_t(y(t,x) - y(t,x'))\| \leqslant L\left(\mathrm{d}_\Omega(x,x') + \|y(t,x) - y(t,x')\|\right)$$

and therefore

$$\|y(t,x) - y(t,x')\| \leqslant e^{tL}\left(\|y^0(x) - y^0(x')\| + \mathrm{d}_\Omega(x,x')\right)$$

Hence $y(t)$ has the same regularity (continuity or Lipschitz) as y^0 and

$$\mathrm{Lip}(y(t,\cdot)) \leqslant e^{tL}\left(1 + \mathrm{Lip}(y(0,\cdot))\right)$$

From micro to macro: graph limit

Set $r_i^N(t) = y(t, x_i^N) - \xi_i^N(t)$, for $i = 1, \ldots, N$. We have

$$\dot{r}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \Big(G(t, x_i^N, x_j^N, y(t, x_i^N), y(t, x_j^N)) - G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)) \Big) + \epsilon_i^N(t)$$

$r_i^N(0) = 0$, with

$$\epsilon_i^N(t) = \int_\Omega G(t, x_i^N, x', y(t, x_i^N), y(t, x'))\, d\nu(x') - \frac{1}{N} \sum_{j=1}^{N} G(t, x_i^N, x_j^N, y(t, x_i^N), y(t, x_j^N))$$

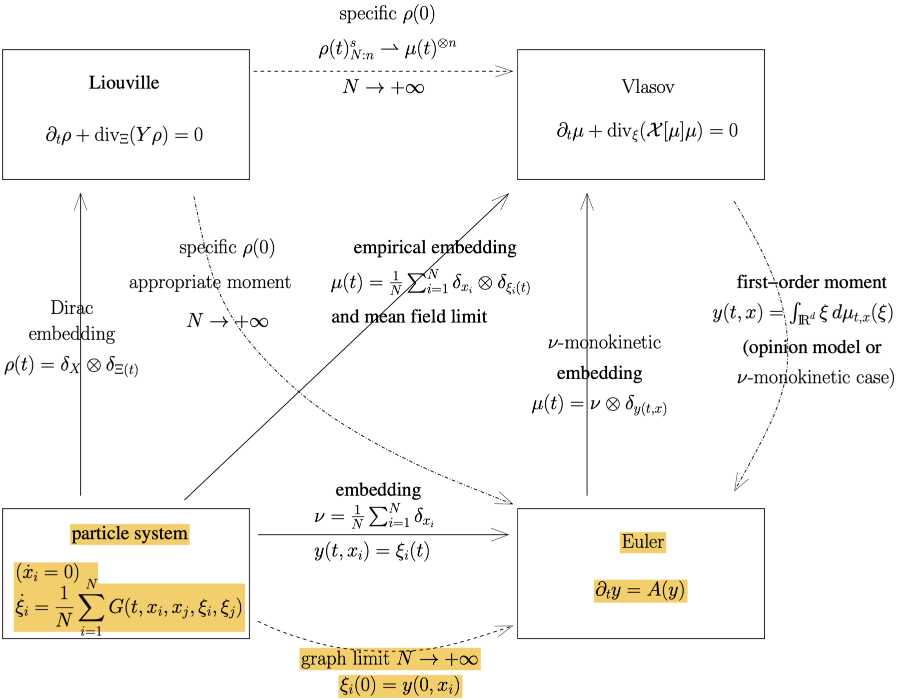
discrepancy between integral and Riemann sum, estimated by

$$\|\epsilon_i^N(t)\| \leqslant \frac{C_\Omega}{N^r} \operatorname{Lip}(x' \mapsto G(t, x_i^N, x', y(t, x_i^N), y(t, x')))$$

Finally, setting $R^N(t) = (r_1^N(t), \ldots, r_N^N(t))$, we get

$$\frac{d}{dt} \|R^N(t)\|_\infty \leqslant \|\dot{R}^N(t)\|_\infty \leqslant L \left(2\|R^N(t)\|_\infty + \frac{C_\Omega}{N^r} (1 + e^{tL}(\operatorname{Lip}(y^0) + 1)) \right)$$

and the theorem easily follows.



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graph limit $N \to +\infty$
$\xi_i(0) = y(0, x_i)$

Toolbox

For measures on $\mathbb{R}^d$:

Wasserstein distance:

$$\forall \mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}^d) \qquad W_1(\mu_1, \mu_2) = \sup \left\{ \int_{\mathbb{R}^d} f \, d(\mu_1 - \mu_2) \;\middle|\; f \in \mathrm{Lip}(\mathbb{R}^d),\ \mathrm{Lip}(f) \leqslant 1 \right\}$$

and more generally $W_p(\mu_1, \mu_2)$ defined with couplings, for every $p \geqslant 1$.

$$W_p(\mu_1, \mu_2) = \inf \left\{ \left(\int_{E^2} \mathrm{d}_E(y_1, y_2)^p \, d\Pi(y_1, y_2) \right)^{1/p} \;\middle|\; \Pi \in \mathcal{P}(E^2),\ (\pi_1)_* \Pi = \mu_1,\ (\pi_2)_* \Pi = \mu_2 \right\}$$

where $\pi_1 : E^2 \to E$ and $\pi_2 : E^2 \to E$ are the canonical projections defined by $\pi_1(y_1, y_2) = y_1$ and $\pi_2(y_1, y_2) = y_2$ for all $(y_1, y_2) \in E \times E$. Equivalently,

$$W_p(\mu_1, \mu_2) = \inf \left\{ \left(\mathbb{E} \, \mathrm{d}_E(Y_1, Y_2)^p \right)^{1/p} \;\middle|\; \mathrm{law}(Y_1) = \mu_1,\ \mathrm{law}(Y_2) = \mu_2 \right\}$$

where the infimum is taken over all possible random variables Y_1 and Y_2 (defined on a same probability space, with values in E) having the laws μ_1 and μ_2 respectively.

Toolbox

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and more generally $W_p(\mu_1, \mu_2)$ defined with couplings, for every $p \geqslant 1$.

For measures on $\Omega \times \mathbb{R}^d$:

Marginal: $\forall \mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)$, its marginal $\nu \in \mathcal{P}(\Omega)$ on Ω is

$$\nu = \pi_* \mu = \mu \circ \pi^{-1} \qquad \text{where} \qquad \pi : \Omega \times \mathbb{R}^d \to \Omega$$

Disintegration of μ wrt ν: $\mu = \int_\Omega \mu_x \, d\nu(x)$ where $\mu_x \in \mathcal{P}(\mathbb{R}^d)$

$L^1_\nu W_p$ distance: $\forall \mu^1, \mu^2 \in \mathcal{P}_p(\Omega \times \mathbb{R}^d)$ having the same marginal ν on Ω, we define

$$(W_p(\mu^1, \mu^2) \leqslant) \qquad L^1_\nu W_p(\mu^1, \mu^2) = \int_\Omega W_p(\mu^1_x, \mu^2_x) \, d\nu(x)$$

Lagrangian viewpoint: mean field limit and Vlasov equation

Mean field

$$\mathcal{X}[\mu](t,x,\xi) = \int_{\Omega\times\mathbb{R}^d} G(t,x,x',\xi,\xi')\,d\mu(x',\xi')$$
$$= \int_{\Omega}\int_{\mathbb{R}^d} G(t,x,x',\xi,\xi')\,d\mu_{x'}(\xi')\,d\nu(x') \qquad \forall (t,x,\xi)\in\mathbb{R}\times\Omega\times\mathbb{R}^d$$

Examples:

Hegselmann–Krause model $\quad \mathcal{X}[\mu](t,x,\xi) = \int_{\Omega\times\mathbb{R}^d} \sigma(x,x')(\xi'-\xi)\,d\mu(x',\xi')$

Cucker–Smale model $\quad \mathcal{X}[\mu](t,x,\xi) = \begin{pmatrix} p \\ \int_{\Omega\times\mathbb{R}^r\times\mathbb{R}^r} a(\|q-q'\|)(p'-p)\,d\mu(x',\xi') \end{pmatrix}$

Lagrangian viewpoint: mean field limit and Vlasov equation

Mean field

$$\begin{aligned}\mathcal{X}[\mu](t,x,\xi) &= \int_{\Omega\times\mathbb{R}^d} G(t,x,x',\xi,\xi')\,d\mu(x',\xi') \\ &= \int_\Omega\int_{\mathbb{R}^d} G(t,x,x',\xi,\xi')\,d\mu_{x'}(\xi')\,d\nu(x') \qquad \forall (t,x,\xi)\in\mathbb{R}\times\Omega\times\mathbb{R}^d\end{aligned}$$

Vlasov equation

$$\partial_t\mu + \mathrm{div}_\xi(\mathcal{X}[\mu]\mu) = 0$$

Equivalently, disintegrating $\mu_t = \mu(t)$ as $\mu_t = \int_\Omega \mu_{t,x}\,d\nu(x)$:

$$\partial_t\mu_{t,x} + \mathrm{div}_\xi(\mathcal{X}[\mu_t](t,x,\cdot)\mu_{t,x}) = 0 \qquad \text{for } \nu\text{-almost every } x\in\Omega$$

Recall that $\mathrm{div}(\mathcal{X}\mu) = L_{\mathcal{X}}\mu$ (Lie derivative of the measure μ) is the measure defined by

$$\langle L_{\mathcal{X}}\mu, f\rangle = -\langle \mu, L_{\mathcal{X}}f\rangle = -\int_{\mathbb{R}^d} \mathcal{X}.\nabla f\,d\mu \qquad \forall f\in\mathscr{C}_c^\infty(\mathbb{R}^d).$$

Lagrangian viewpoint: mean field limit and Vlasov equation

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Vlasov equation

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Concept of solution:

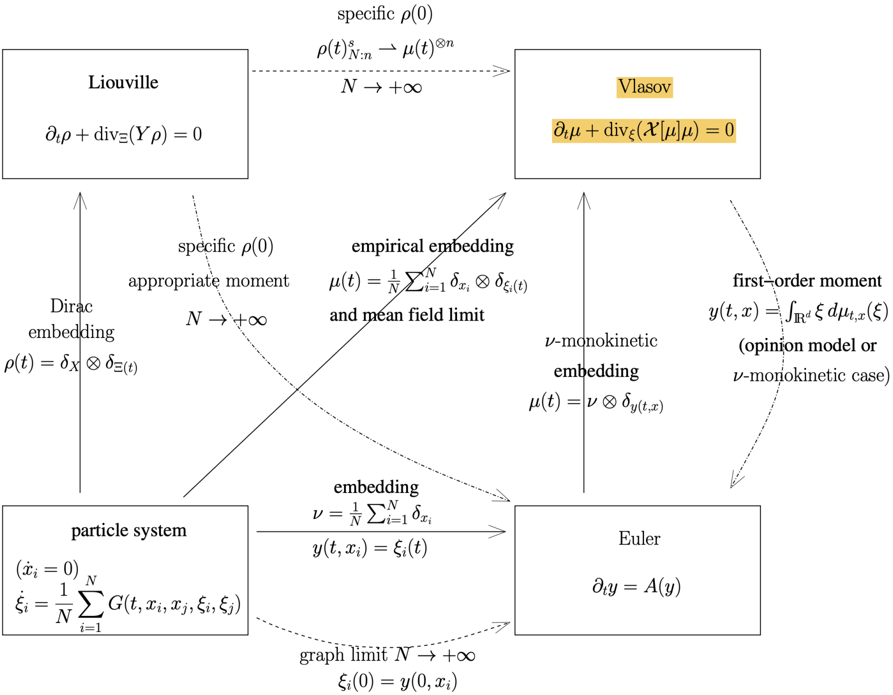
- $\mathscr{C}^0_{\mathrm{comp}}([0,T),\mathcal{P}_c(\Omega\times\mathbb{R}^d))$ = set of $\mu\in\mathscr{C}^0([0,T),\mathcal{P}_c(\Omega\times\mathbb{R}^d))$ that are equi-compactly supported on any compact interval of $[0,T)$, i.e.:

$$\forall t_1\in(0,T) \quad \exists K\subset\Omega\times\mathbb{R}^d \quad | \quad \mathrm{supp}(\mu(t))\subset K \quad \forall t\in[0,t_1].$$

- A solution $t\mapsto\mu(t)$ of Vlasov on $[0,T)$ such that $\mu(0)=\mu_0\in\mathcal{P}_c(\Omega\times\mathbb{R}^d)$ is a $\mu\in\mathscr{C}^0_{\mathrm{comp}}([0,T),\mathcal{P}_c(\Omega\times\mathbb{R}^d))$ s.t. $\forall g\in C_c^\infty(\Omega\times\mathbb{R}^d)$, $t\mapsto\int g\,d\mu_t$ is AC on $[0,T)$ and

$$\int_{\Omega\times\mathbb{R}^d} g\,d\mu_t = \int_{\Omega\times\mathbb{R}^d} g\,d\mu_0 + \int_0^t\int_{(\Omega\times\mathbb{R}^d)^2}\langle\nabla_\xi g(x,\xi),G(\tau,x,x',\xi,\xi')\rangle\,d\mu_\tau(x',\xi')\,d\mu_\tau(x,\xi)\,d\tau$$

a.e. on $[0,T)$.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Lagrangian viewpoint: mean field limit and Vlasov equation

Theorem: existence, uniqueness and stability for Vlasov

1. $\forall \mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ $\exists! \mu \in \mathscr{C}^0([0, T_0), \mathcal{P}_c(\Omega \times \mathbb{R}^d))$ (with $T_0 = T_{\max}(\mathrm{supp}(\mu_0))$) solution of Vlasov s.t. $\mu(0) = \mu_0$, locally Lipschitz / t for the distance W_p.

   We have

   $$\boxed{\mu(t) = \varphi_{\mu_0}(t)_* \mu_0}$$

   meaning that $\mu_{t,x} = \varphi_{\mu_0}(t, x, \cdot)_* \mu_{0,x} \quad \forall t \in [0, T_0)$ and ν-a.e. $x \in \Omega$, where $t \mapsto \varphi_{\mu_0}(t, x, \cdot)$ is the unique solution (Vlasov flow) of

   $$\partial_t \varphi_{\mu_0}(t, x, \cdot) = \mathcal{X}[\mu(t)](t, x, \cdot) \circ \varphi_{\mu_0}(t, x, \cdot)$$
   $$\varphi_{\mu_0}(0, x, \cdot) = \mathrm{id}_{\mathbb{R}^d} \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

   Moreover, if $\mu_0 \in \mathcal{P}_c^{ac}(\Omega \times \mathbb{R}^d)$ then $\mu(t) \in \mathcal{P}_c^{ac}(\Omega \times \mathbb{R}^d)$ for every $t \in [0, T_0)$.

Lagrangian viewpoint: mean field limit and Vlasov equation

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   Moreover:

   1.1. For equi-compactly supported sequences:

   $$W_p(\mu^k(0), \mu(0)) \to 0 \;\Rightarrow\; W_p(\mu^k(t), \mu(t)) \to 0.$$

   1.2. For all solutions μ^1, μ^2 on $[0, T]$ of Vlasov having the same marginal ν,

   $$L^1_\nu W_p(\mu^1(t), \mu^2(t)) \leqslant C\, e^{t\, \mathrm{Lip}_{\xi,\xi'}(G)}\, L^1_\nu W_p(\mu^1(0), \mu^2(0)) \qquad \forall t \in [0, T]$$

2. If G is locally Lipschitz / (x, x', ξ, ξ') then for all solutions μ^1, μ^2 of Vlasov,

   $$W_p(\mu^1(t), \mu^2(t)) \leqslant C\, e^{t\, \mathrm{Lip}_{x,x'\,\xi,\xi'}(G)}\, W_p(\mu^1(0), \mu^2(0)) \qquad \forall t \in [0, T]$$

   (classical Dobrushin estimate) Lipschitz constants estimated on the supports of μ^1, μ^2.

Lagrangian viewpoint: mean field limit and Vlasov equation

Relation between Vlasov and the particle system: as usual with empirical measures

$$\mu^e_{(X^N,\Xi^N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N} \otimes \delta_{\xi_i^N}$$

with $X^N = (x_1^N, \ldots, x_N^N) \in \Omega^N$ and $\Xi^N = (\xi_1^N, \ldots, \xi_N^N) \in \mathbb{R}^{dN}$.

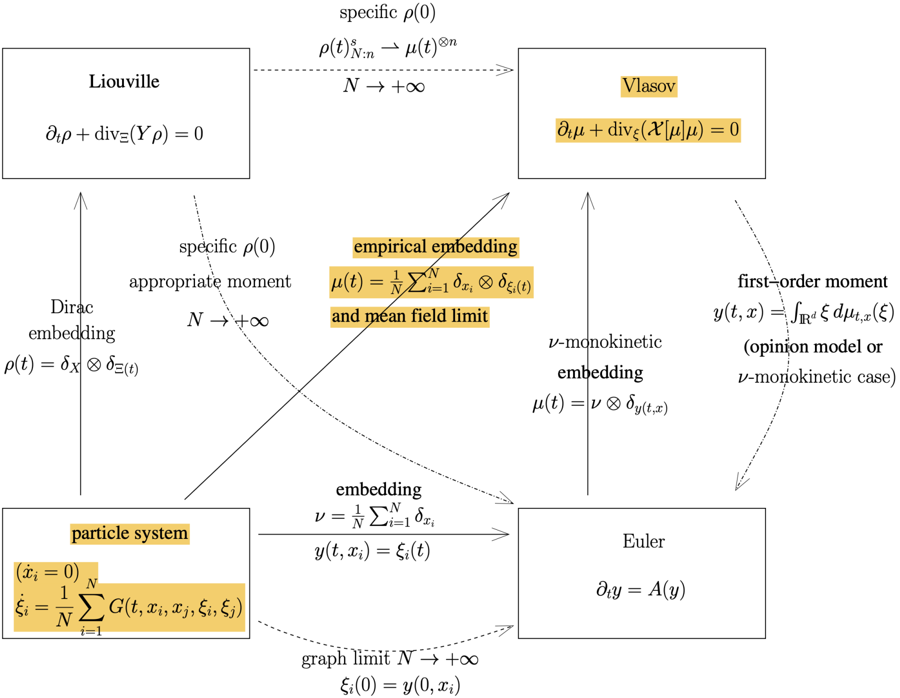
Proposition

$t \mapsto \Xi^N(t)$ with $X^N \in \Omega^N$ solution of the particle system $\Rightarrow t \mapsto \mu^e_{(X^N,\Xi^N(t))}$ solution of Vlasov.
Converse true if all x_i^N and all $\xi_i^N(t)$ are distinct.

Corollary

$W_p\left(\mu^e_{(X^N,\Xi_0^N)}, \mu_0\right) \to 0$ as $N \to +\infty \Rightarrow W_p\left(\mu^e_{(X^N,\Xi^N(t))}, \mu(t)\right) \to 0$ as $N \to +\infty$.

(with estimates if G is locally Lipschitz with respect to (x, x', ξ, ξ'))



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Lagrangian viewpoint: mean field limit and Vlasov equation

Sketch of proof of the theorem:

Given $\mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$, consider a sequence of empirical measures

$$\mu^e_{(X^N,\Xi^N_0)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N} \otimes \delta_{\xi^N_{0,i}} \rightharpoonup \mu_0 \qquad \text{as } N \to +\infty$$

with $(X^N, \Xi^N_0) \in (\mathrm{supp}(\mu_0))^N$, where $X^N = (x^N_1, \dots, x^N_N)$ and $\Xi^N_0 = (\xi^N_1, \dots, \xi^N_N)$.

Let $\Xi^N(t) = (\xi^N_1(t), \dots, \xi^N_N(t))$ be the solution of the particle system s.t. $\Xi^N(0) = \Xi^N_0$. It is defined on $[0, T]$ for every $T \in (0, T_{\max}(\mathrm{supp}(\mu_0)))$. Then

$$\mu^e_{(X^N,\Xi^N(t))} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N} \otimes \delta_{\xi^N_i(t)}$$

is solution on $[0, T]$ of Vlasov $\partial_t \mu + L_{\mathcal{X}[\mu]}\mu = 0$, and

$$\exists K \subset \Omega \times \mathbb{R}^d \text{ compact s.t. } \quad \mathrm{supp}(\mu^e_{(X^N,\Xi^N(t))}) \subset K \quad \forall t \in [0, T] \quad \forall N \in \mathbb{N}^*$$

Up to subsequence: $\mu^e_{(X^N,\Xi^N(\cdot))} \rightharpoonup \mu \in L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$ in weak star topology.

Lagrangian viewpoint: mean field limit and Vlasov equation

By classical functional analysis arguments:

Lemma

Let $K \subset \Omega \times \mathbb{R}^d$ *compact,* $\mu_0 \in \mathcal{P}_c(K)$ *and* $T > 0$*. Consider a sequence of solutions* $\mu^k \in \mathscr{C}^0([0,T], \mathcal{P}_c(K))$ *of Vlasov* $\partial_t \mu^k + L_{\mathcal{X}[\mu^k]}\mu^k = 0$ *such that, as* $k \to +\infty$*:*

- $\mu^k(0)$ *converges weakly to* μ_0*,*
- μ^k *converges to* $\mu \in L^\infty([0,T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$ *for the weak star topology.*

Then $\mu \in C^0([0,T], \mathcal{P}_c(K))$ *and* $t \mapsto \mu(t)$ *is Lipschitz in* W_p *distance and is the solution of Vlasov* $\partial_t \mu + L_{\mathcal{X}[\mu]}\mu = 0$ *s.t.* $\mu(0) = \mu_0$*.*
Moreover, $W_p(\mu^k(t), \mu(t)) \to 0$ *uniformly wrt* $t \in [0,T]$*.*

Therefore: we conclude existence (not yet uniqueness).

Lagrangian viewpoint: mean field limit and Vlasov equation

Uniqueness follows from Gronwall type arguments using that:

– On the one part, for all $\mu^1, \mu^2 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$

$$\left\| \mathcal{X}[\mu^1](t,x,\xi) - \mathcal{X}[\mu^2](t,x,\xi) \right\| = \left\| \int_{\Omega \times \mathbb{R}^d} G(t,x,x',\xi,\xi')\, d(\mu^1(x',\xi') - \mu^2(x',\xi')) \right\|$$
$$\leqslant \mathrm{Lip}(G(t,x,\cdot,\xi,\cdot)_{|S})\, W_1(\mu^1,\mu^2)$$

where $S = \mathrm{supp}(\mu^1) \cup \mathrm{supp}(\mu^2)$ (compact set).

And for all $\mu^1, \mu^2 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ *having the same marginal* $\nu \in \mathcal{P}_c(\Omega)$ on Ω,

$$\left\| \mathcal{X}[\mu^1](t,x,\xi) - \mathcal{X}[\mu^2](t,x,\xi) \right\| = \left\| \int_{\Omega} \int_{\mathbb{R}^d} G(t,x,x',\xi,\xi')\, d(\mu^1_{x'}(\xi') - \mu^2_{x'}(\xi'))\, d\nu(x') \right\|$$
$$\leqslant \max_{x' \in \mathrm{supp}(\nu)} \mathrm{Lip}(G(t,x,x',\xi,\cdot)_{|S_{x'}})\, L^1_\nu W_1(\mu^1,\mu^2)$$

where $S_{x'} = \mathrm{supp}(\mu^1_{x'}) \cup \mathrm{supp}(\mu^2_{x'})$ (compact) and $L^1_\nu W_1(\mu^1,\mu^2) = \int_\Omega W_1(\mu^1_{x'},\mu^2_{x'})\, d\nu(x')$.

Lagrangian viewpoint: mean field limit and Vlasov equation

– On the other part:

Lemma (Propagation)

For $i = 1, 2$, let $Y^i(t, \lambda, \cdot)$ be a continuous time-varying vector field on E (Banach), depending on the parameter $\lambda \in \Lambda$ (Polish space), locally Lipschitz with respect to $(\lambda, y) \in \Lambda \times E$ uniformly with respect to t on any compact interval, generating a flow:

$$\partial_t \Phi^i(t, t_0, \lambda, y) = Y^i(t, \lambda, \Phi^i(t, t_0, \lambda, y))$$
$$\Phi^i(t_0, t_0, \lambda, y) = y \qquad \forall t_0 \in \mathbb{R},\ y \in E,\ \lambda \in \Lambda.$$

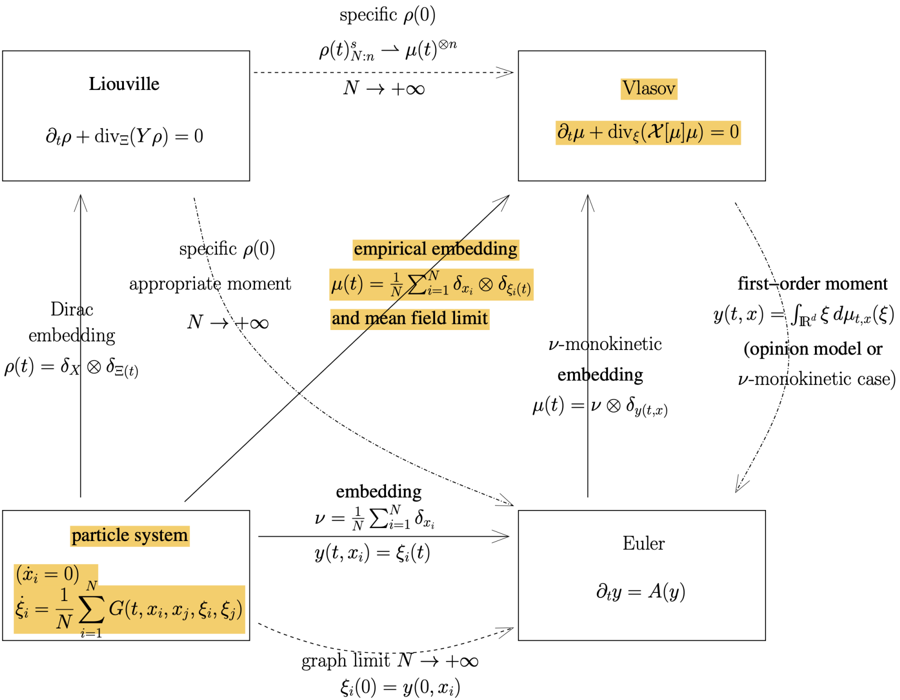
Given any $\mu^1(t_0), \mu^2(t_0) \in \mathcal{P}_c(\Lambda \times E)$, set $\mu^i_t = \mu^i(t) = \Phi^i(t, t_0)_* \mu^i(t_0)$. Then:

$$W_p(\mu^1(t), \mu^2(t)) \leqslant e^{(t-t_0)L([t_0,t])} W_p(\mu^1(t_0), \mu^2(t_0)) + M([t_0, t]) \frac{e^{(t-t_0)L([t_0,t])} - 1}{L([t_0, t])}$$

where $L([t_0, t]) = \max_{t_0 \leqslant \tau \leqslant t} \mathrm{Lip}\left(Y^1(\tau, \cdot, \cdot)_{|S(\tau)}\right)$,

$$S(t) = (\mathrm{supp}(\nu^1) \cup \mathrm{supp}(\nu^2)) \times \Phi^1(t, t_0, \mathrm{supp}(\mu^1(t_0)) \cup \mathrm{supp}(\mu^2(t_0))) \ \cup\ \mathrm{supp}(\mu^2(t)),$$

$$M([t_0, t]) = \max\{\| Y^1(\tau, \lambda, y) - Y^2(\tau, \lambda, y)\|_E \ \mid\ t_0 \leqslant \tau \leqslant t,\ (\lambda, y) \in \mathrm{supp}(\mu^2(\tau))\}.$$



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Eulerian viewpoint: Liouville equation

Recall that the particle system

$$\dot{\xi}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \qquad i = 1, \dots, N$$

is equivalently written as

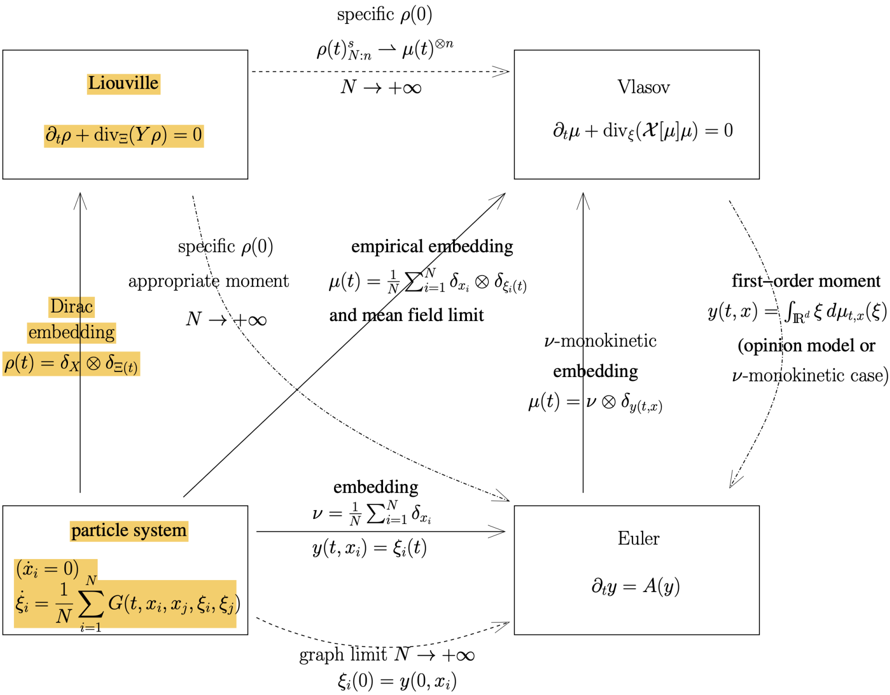
$$\dot{\Xi}^N(t) = Y^N(t, X^N, \Xi^N(t)) = \left(Y_1^N(t, X^N, \Xi^N(t)), \dots, Y_N^N(t, X^N, \Xi^N(t))\right)$$

with

$$X^N = (x_1^N, \dots, x_N^N), \qquad \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$$

$$Y_i^N(t, X, \Xi) = \frac{1}{N}\sum_{j=1}^{N} G(t, x_i, x_j, \xi_i, \xi_j)$$

Recall that the particle flow $(\Phi^N(t, X, \cdot))_{t \in I}$ ($I \subset \mathbb{R}$) is the local-in-time flow of diffeomorphisms of $\mathbb{R}^{dN}$ generated by the time-dependent vector field $Y^N(t, X, \cdot)$.



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Eulerian viewpoint: Liouville equation

Proposition

Solutions $\rho \in \mathscr{C}^0([0,T], \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN}))$ of the (N-body) Liouville equation

$$\boxed{\partial_t \rho + \mathrm{div}_\Xi(Y\rho) = 0}$$

(usual transport equation on $\mathbb{R}^{dN}$) are given by pushforward under the particle flow:

$$\boxed{\rho(t) = \Phi(t)_* \rho(0)}$$

Embedding particles to Liouville

$t \mapsto \Xi^N(t)$ solution of the particle system $\Leftrightarrow$ $t \mapsto \rho^N(t) = \delta_{X^N} \otimes \delta_{\Xi^N(t)}$ solution of Liouville.

Probabilistic interpretation: while $\Xi^N(t)$ (particle) is deterministic, $\rho_t^N(X,\Xi)$ is the probability that at time t each particle i be at (x_i, ξ_i), $i = 1, \ldots, N$.

Here: ρ_t^N = probability on the big space $(\Omega \times \mathbb{R}^d)^N$.

$\neq$ mean field limit ($\mu(t)$ = probability measure on $\Omega \times \mathbb{R}^d$) in which we take the limit of the average over all particles but one.

Recovering Vlasov from Liouville by taking marginals

Objective: search for a relationship between $\mu(t)$ and $\rho^N(t)$ by taking marginals of $\rho^N(t)$.

(cf Jabin 2014 and Golse Mouhot Paul 2016 in quantum mechanics)

Let $\mu_0 = \int_\Omega \mu_{0,x}\, d\nu(x) \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$. We take

(i) either $\rho_0^N = \delta_{X^N} \otimes \delta_{\Xi_0^N}$ (Dirac) s.t. $\mu^e_{(X^N,\Xi_0^N)} \rightharpoonup \mu_0$,

(ii) or $\rho_0^N = \delta_{x_1^N} \otimes \cdots \delta_{x_N^N} \otimes \mu_{0,x_1^N} \otimes \cdots \otimes \mu_{0,x_N^N}$ (semi-Dirac) assuming that $x \mapsto \mu_{0,x}$ is ν-a.e. continuous for W_1.

Let $\rho^N(t)^s_{N:k}$ be the k^{th}-order marginal of the symmetrization of $\rho^N(t)$.

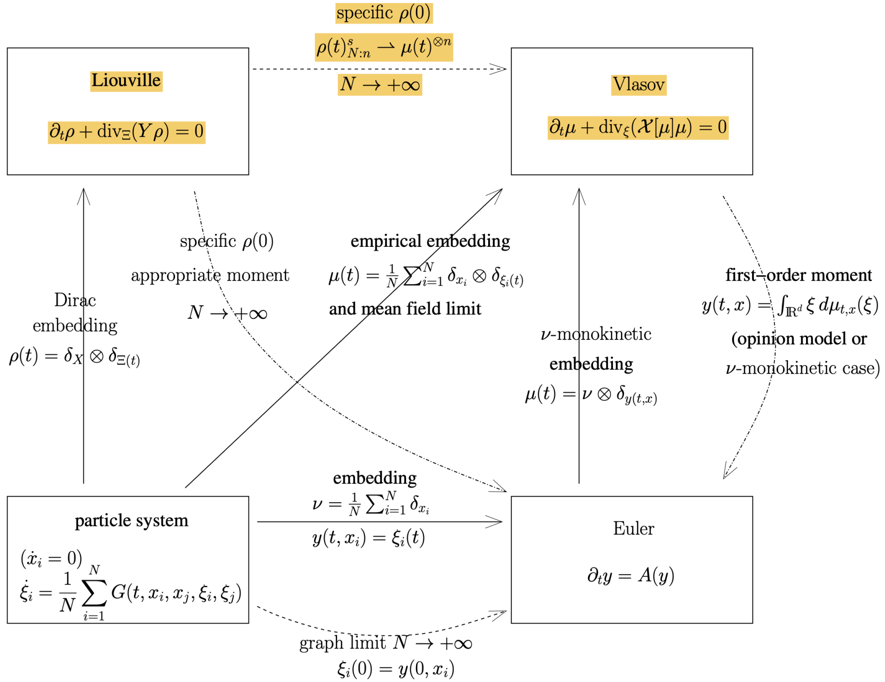
Theorem (propagation of chaos)

1. $\forall k \in \mathbb{N}^*$ $\quad W_p\big(\rho(t)^s_{N:k}, \mu(t)^{\otimes k}\big) \to 0$ as $N \to +\infty$.
2. If moreover G is locally Lipschitz / (x, x', ξ, ξ') and, in case (ii), $x \mapsto \mu_{0,x}$ is Lipschitz then

$$W_p\big(\rho(t)^s_{N:k}, \mu(t)^{\otimes k}\big) \leqslant C\, e^{t\,\mathrm{Lip}(G)} \frac{1}{N^{r/p}} \qquad \forall k \in \{1, \dots, N^{(1-r)/2}\}$$

with $r = 1/(n+d)$ if Ω is a n-dimensional manifold (Lipschitz constant estimated on the supports)

The proof uses some combinatorial arguments combined with Wasserstein estimates.


specific $\rho(0)$
$\rho(t)^s_{N:n} \rightharpoonup \mu(t)^{\otimes n}$
$N \to +\infty$
Liouville
$\partial_t \rho + \mathrm{div}_\Xi(Y\rho) = 0$
Vlasov
$\partial_t \mu + \mathrm{div}_\xi(\mathcal{X}[\mu]\mu) = 0$
specific $\rho(0)$
appropriate moment
$N \to +\infty$
empirical embedding
$\mu(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \otimes \delta_{\xi_i(t)}$
and mean field limit
Dirac
embedding
$\rho(t) = \delta_X \otimes \delta_{\Xi(t)}$
ν-monokinetic
embedding
$\mu(t) = \nu \otimes \delta_{y(t,x)}$
first–order moment
$y(t,x) = \int_{\mathbb{R}^d} \xi \, d\mu_{t,x}(\xi)$
(opinion model or
ν-monokinetic case)
embedding
$\nu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$
$y(t,x_i) = \xi_i(t)$
particle system
$(\dot{x}_i = 0)$
$\dot{\xi}_i = \frac{1}{N}\sum_{i=1}^N G(t, x_i, x_j, \xi_i, \xi_j)$
Euler
$\partial_t y = A(y)$
graph limit $N \to +\infty$
$\xi_i(0) = y(0, x_i)$

From Vlasov to Euler: hydrodynamic limit

Hydrodynamic limit: (see Spohn 1991)

Given any $\mu = \int_\Omega \mu_x \, d\nu(x) \in \mathcal{P}(\Omega \times \mathbb{R}^d)$, the three macroscopic quantities that are usually considered in the hydrodynamic limit procedure are the three first moments of the measure μ with respect to ξ:

- (order 0) total mass $\rho(x) \geqslant 0$ of μ_x:

$$\rho(x) = \int_{\mathbb{R}^d} d\mu_x(\xi) = \mu_x(\mathbb{R}^d) = 1 \qquad \text{for } \nu\text{-a.e. } x \in \Omega$$

  (uninteresting here)

- (order 1) “speed” $y(x) \in \mathbb{R}^d$:

$$\rho(x)y(x) = \int_{\mathbb{R}^d} \xi \, d\mu_x(\xi)$$

  (expectation of any random law of probability distribution μ_x)

- (order 2) “temperature” $T(x) \geqslant 0$:

$$d\rho(x)T(x) = \int_{\mathbb{R}^d} \|\xi - y(x)\|^2 \, d\mu_x(\xi)$$

  (variance)

From Vlasov to Euler: hydrodynamic limit

Given any solution $\mu(\cdot)$ of Vlasov, we define its moment of order 1 ("speed") by

$$\boxed{y(t,x) = \int_{\mathbf{R}^d} \xi \, d\mu_{t,x}(\xi)}$$

Using the Vlasov equation, we have

$$\begin{aligned}
\partial_t y(t,x) &= \langle \partial_t \mu_{t,x}, \xi \mapsto \xi \rangle \\
&= \big\langle \mu_{t,x}, L_{\mathcal{X}[\mu_t](t,x,\cdot)}(\xi \mapsto \xi) \big\rangle \\
&= \int_{\mathbf{R}^d} \mathcal{X}[\mu_t](t,x,\xi) \, d\mu_{t,x}(\xi) \\
&= \int_{\mathbf{R}^d} \int_{\Omega \times \mathbf{R}^d} G(t,x,x',\xi,\xi') \, d\mu_t(x',\xi') \, d\mu_{t,x}(\xi).
\end{aligned}$$

(kind of "mean" mean field, since the mean field is now averaged under $\mu_{t,x}$)

From Vlasov to Euler: hydrodynamic limit

Given any solution $\mu(\cdot)$ of Vlasov, we define its moment of order 1 ("speed") by

$$\boxed{y(t,x) = \int_{\mathbf{R}^d} \xi \, d\mu_{t,x}(\xi)}$$

Consequence:

Hegselmann–Krause model: linear Euler equation

When $G(t,x,x',\xi,\xi') = \sigma(x,x')(\xi' - \xi)$ we have the Euler equation

$$\boxed{\partial_t y(t,x) = Ay(t,x) = \int_\Omega \sigma(x,x')(y(t,x') - y(t,x)) \, d\nu(x')}$$

Proof:

$$\partial_t y(t,x) = \underbrace{\int_{\mathbf{R}^d} d\mu_{t,x}(\xi)}_{=1} \int_\Omega \sigma(x,x') \underbrace{\int_{\mathbf{R}^d} \xi' \, d\mu_{t,x'}(\xi')}_{=y(t,x')} d\nu(x')$$
$$- \underbrace{\int_{\mathbf{R}^d} \xi \, d\mu_{t,x}(\xi)}_{=y(t,x)} \int_\Omega \sigma(x,x') \underbrace{\int_{\mathbf{R}^d} d\mu_{t,x'}(\xi')}_{=1} d\nu(x')$$

From Vlasov to Euler: hydrodynamic limit

Given any solution $\mu(\cdot)$ of Vlasov, we define its moment of order 1 (“speed”) by

$$\boxed{y(t,x) = \int_{\mathbb{R}^d} \xi \, d\mu_{t,x}(\xi)}$$

In the general case, no closed equation (hierarchy of coupled moments).

Given ν, we define the ν*-monokinetic* measure

$$\mu_y^\nu = \nu \otimes \delta_{y(\cdot)}$$

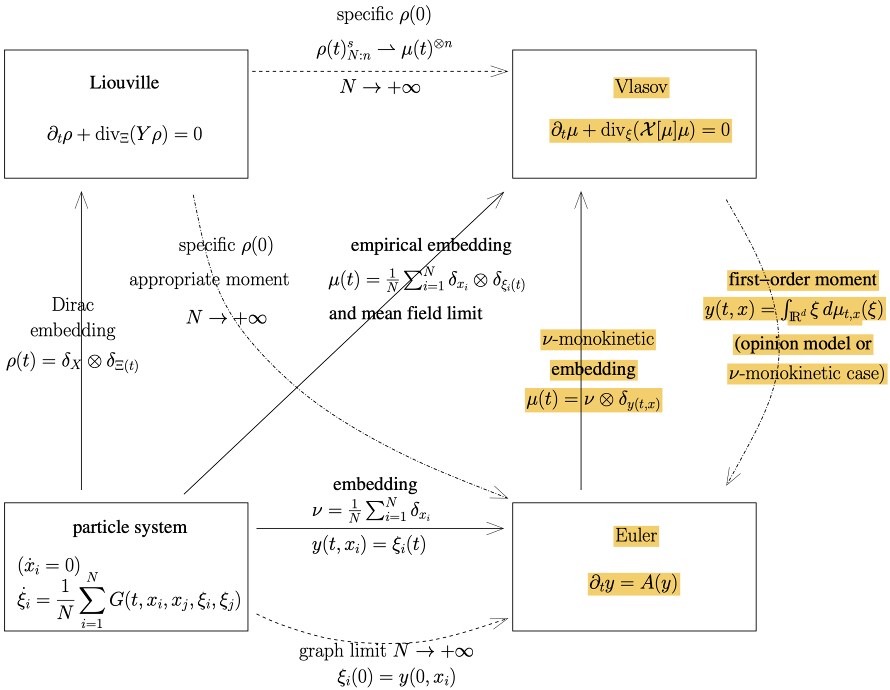
Monokinetic case

$t \mapsto \mu(t) = \mu^\nu_{y(t,\cdot)} \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ is solution of Vlasov

$\Leftrightarrow t \mapsto y(t,\cdot) \in L^\infty_\nu(\Omega, \mathbb{R}^d)$ is solution of the (nonlinear) Euler equation

$$\boxed{\partial_t y(t,x) = A(t, y(t,x)) = \int_\Omega G(t,x,x',y(t,x),y(t,x'))\, d\nu(x')}$$

Indeed, when $\mu_t = \mu^\nu_{y(t,\cdot)}$, we have $\mathcal{X}[\mu_t](t,x,\xi) = \int_\Omega G(t,x,x',\xi,y(t,x'))\, d\nu(x')$.



specific $\rho(0)$
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Euler
$\partial_t y = A(y)$
graph limit $N \to +\infty$
$\xi_i(0) = y(0, x_i)$

From Vlasov to Euler: hydrodynamic limit

Moment of order 2 ("temperature"):

$$\boxed{T(t,x) = \frac{1}{d}\int_{\mathbf{R}^d} \|\xi - y(t,x)\|^2 \, d\mu_{t,x}(\xi) \qquad \forall x \in \Omega}$$

Assume that $\|\cdot\|$ is Euclidean.

$$\partial_t T(t,x) = \frac{1}{d}\left\langle \partial_t \mu_{t,x}, \xi \mapsto \|\xi - y(t,x)\|^2 \right\rangle - \underbrace{\frac{2}{d}\left\langle \mu_{t,x}, \langle \xi - y(t,x), \partial_t y(t,x)\rangle_{\mathbf{R}^d} \right\rangle}_{=0}$$

$$= \frac{2}{d}\left\langle \mu_{t,x}, \xi \mapsto \langle \xi - y(t,x), \mathcal{X}[\mu_t](t,x,\xi)\rangle_{\mathbf{R}^d} \right\rangle$$

$$= \frac{2}{d}\int_{\mathbf{R}^d} \langle \xi - y(t,x), \mathcal{X}[\mu_t](t,x,\xi)\rangle_{\mathbf{R}^d} \, d\mu_{t,x}(\xi)$$

From Vlasov to Euler: hydrodynamic limit

Moment of order 2 ("temperature"):

$$T(t,x) = \frac{1}{d}\int_{\mathbf{R}^d} \|\xi - y(t,x)\|^2 \, d\mu_{t,x}(\xi) \qquad \forall x \in \Omega$$

Assume that $\|\cdot\|$ is Euclidean.

$$\partial_t T(t,x) = \frac{2}{d}\int_{\mathbf{R}^d} \langle \xi - y(t,x), \mathcal{X}[\mu_t](t,x,\xi)\rangle_{\mathbf{R}^d} \, d\mu_{t,x}(\xi)$$

In the Hegselmann–Krause model, setting $S(x) = \int_\Omega \sigma(x,x')\, d\nu(x')$:

$$\begin{aligned}
\mathcal{X}[\mu_t](t,x,\xi) &= \int_\Omega \int_{\mathbf{R}^d} \sigma(x,x')(\xi' - \xi)\, d\mu_{t,x'}(\xi')\, d\nu(x') \\
&= -\int_\Omega \int_{\mathbf{R}^d} \sigma(x,x')(\xi - y(t,x))\, d\mu_{t,x'}(\xi')\, d\nu(x') \\
&\qquad\qquad + \int_\Omega \int_{\mathbf{R}^d} \sigma(x,x')(\xi' - y(t,x))\, d\mu_{t,x'}(\xi')\, d\nu(x') \\
&= -S(x) \int_{\mathbf{R}^d} (\xi - y(t,x))\, d\mu_{t,x'}(\xi') \ + \ F(t,x) \text{ not depending on } \xi
\end{aligned}$$

Hence:

From Vlasov to Euler: hydrodynamic limit

Moment of order 2 (“temperature”):

$$\boxed{T(t,x) = \frac{1}{d}\int_{\mathbb{R}^d} \|\xi - y(t,x)\|^2 \, d\mu_{t,x}(\xi) \qquad \forall x \in \Omega}$$

Assume that $\|\cdot\|$ is Euclidean.

Hegselmann–Krause model

In the Hegselmann–Krause model $G(t,x,x',\xi,\xi') = \sigma(x,x')(\xi'-\xi)$ we have

$$\boxed{\partial_t T(t,x) = -2S(x)T(t,x)} \qquad \text{where} \qquad S(x) = \int_\Omega \sigma(x,x')\, d\nu(x')$$

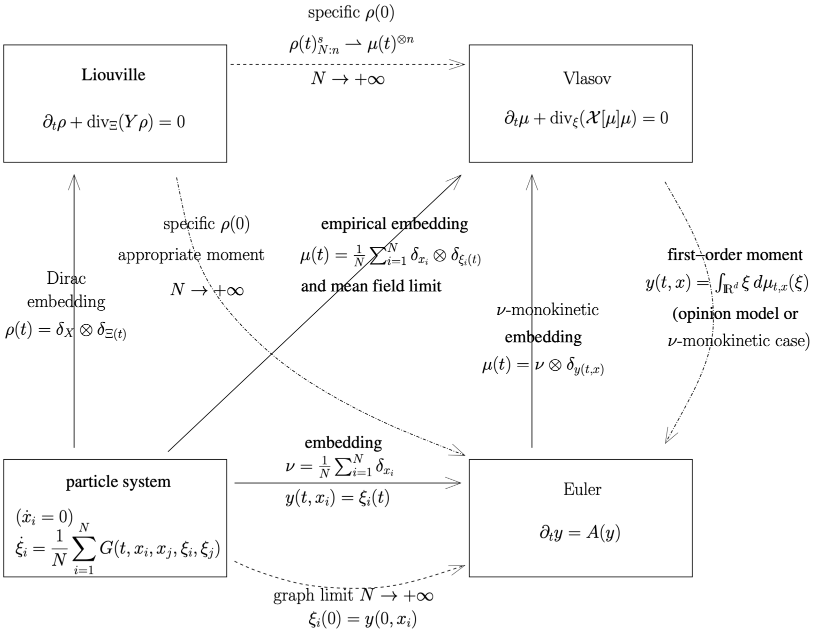
Hence $t \mapsto T(t,x) = T(0,x)e^{-2tS(x)}$ decreases exponentially to 0 as $t \to +\infty$ for ν-almost every $x \in \Omega$ such that $S(x) > 0$.

Actually: same result for all moments of order $\geqslant 2$.

$\Rightarrow$ slight generalization of [Boudin Salvarani Trélat, SIMA 2022] (convergence to consensus).

In general: open problem of how to close the coupled moment equations.

Summary: relationships between particle (microscopic) system, Liouville (probabilistic) equation, Vlasov (mesoscopic, mean field) equation, Euler (macroscopic, graph limit) eq.



A surprising consequence:

finite particle approximation of quasilinear PDEs

Particle approximation of any linear PDE: main idea

$$\boxed{\partial_t y = Ay}$$

with $A : D(A) \to L^2(\Omega)$ generating a C_0-semigroup. Two steps:

1. Approximate A with a bounded operator A_ε, given by $(A_\varepsilon f)(x) = \int_\Omega \sigma_\varepsilon(x, x') f(x')\, dx'$ (e.g.: Yosida approximation, or convolution), so that

$$\partial_t y_\varepsilon = A_\varepsilon y_\varepsilon, \quad y_\varepsilon(0) = y(0) \quad \Rightarrow \quad \|y(t) - y_\varepsilon(t)\|_{L^\infty} = \mathrm{O}(\varepsilon)$$

2. Particle approximation: $\displaystyle\int_\Omega \sigma_\varepsilon(x, x') f(x')\, dx' \simeq \frac{1}{N}\sum_{j=1}^{N} \sigma_\varepsilon(x, x_j^N) f(x_j^N)$ leading to the

particle system: $$\boxed{\dot{\xi}_i(t) = \frac{1}{N}\sum_{j=1}^{N} \sigma_\varepsilon(x_i^N, x_j^N)\, \xi_j^N(t)}$$ $\to$ 2 parameters $\varepsilon \to 0$ and $N \to +\infty$

The estimates of the previous results lead to $\displaystyle\left\| y(t,\cdot) - \sum_{i=1}^{N} \xi_{\varepsilon,i}^N(t)\, \mathbb{1}_{\Omega_i^N}(\cdot) \right\|_{L^2} \lesssim \frac{1}{\ln \ln N}$.

Particle approximations of PDEs

Assumptions:

(O_1) Either $\Omega \subset \mathbb{R}^n$ compact Lipschitz domain, d_Ω Euclidean distance, ν Lebesgue;

(O_2) or Ω smooth compact Riemannian manifold of dimension n,
d_Ω Riemannian distance, ν is the canonical Riemannian measure;

and moreover assume that $\nu(\Omega) = 1$.

General quasilinear PDE: $\quad p \in \mathbb{N}^*, \quad a_\alpha \in L^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$,

$$\boxed{\partial_t y(t,x) = \sum_{|\alpha| \leqslant p} a_\alpha(t,x,y(t,x))\, D^\alpha y(t,x) = A(t,y(t,x))\, y(t,x)} \qquad (PDE)$$

with arbitrary conditions at the boundary of Ω in case (O_1), assumed to be well-posed (semi-group, or evolution system).

Objective: design finite particle systems approximating the solutions of (PDE).

Particle approximations of PDEs

<u>Idea:</u> if $G(t,x,x',\xi,\xi') = \sigma(x,x')\xi'$ then

$$\mathcal{X}[\mu](x) = \int_\Omega \sigma(x,x')y(x')\,d\nu(x') = (Ay)(x)$$

$\Rightarrow$ (Hilbert-Schmidt) operator A of kernel σ wrt ν, and Euler equation $\partial_t y = Ay$.

Particle approximations of PDEs

Idea: if $G(t, x, x', \xi, \xi') = \sigma(x, x')\xi'$ then

$$\mathcal{X}[\mu](x) = \int_\Omega \sigma(x, x')y(x')\, d\nu(x') = (Ay)(x)$$

$\Rightarrow$ (Hilbert-Schmidt) operator A of kernel σ wrt ν, and Euler equation $\partial_t y = Ay$.

Reminder: Schwartz kernel theorem

Any linear operator $A : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ has a distributional Schwartz kernel $[A]$:

$$(Af)(x) = \langle [A](x, \cdot), f \rangle_{\mathcal{D}', \mathcal{D}} = \int_\Omega [A](x, x')f(x') \qquad \forall f \in \mathcal{D}(\Omega) \qquad \forall x \in \Omega$$

Example: $\Omega = \mathbb{R}, \quad A = \partial_x, \quad [A](x, y) = -\delta'(x - y)$

$\rightsquigarrow$ Idea: approximate the Schwartz kernel with a smooth function σ_ε.

Particle approximations of PDEs

Take *any* quasilinear operator

$$A(t,\xi)y(x) = \int_\Omega [A](t,x,x',\xi)y(x')$$

of Schwartz kernel $[A](t,\cdot,\cdot,\xi)$.

One can design smooth functions σ_ε approximating $[A]$, and set

$$G_\varepsilon(t,x,x',\xi,\xi') = \sigma_\varepsilon(t,x,x',\xi)\,\xi' \quad \text{and} \quad A_\varepsilon(t,f)(x) = \int_\Omega G_\varepsilon(t,x,x',f(x),f(x'))\,dx'$$

$\Rightarrow$ classical Euler equation

$$\boxed{\partial_t y_\varepsilon(t,x) = \int_\Omega \sigma_\varepsilon(t,x,x',y_\varepsilon(t,x))y_\varepsilon(t,x')\,dx'}$$

and particle system

$$\boxed{\dot\xi^N_{\varepsilon,i}(t) = \frac{1}{N}\sum_{j=1}^N \sigma_\varepsilon\left(t,x_i^N,x_j^N,\xi^N_{\varepsilon,i}(t)\right)\xi^N_{\varepsilon,j}(t), \qquad i=1,\ldots,N}$$

Particle approximations of PDEs

Theorem

Let $T > 0$. Assume that:

- $a_\alpha \in W^{1,\infty}(\Omega)$ and that A generates a semigroup (or evolution system);
- $y \in L^1([0,T], W^{p+1,\infty}(\Omega, \mathbb{R}^d))$ is a solution of (PDE) s.t. $y(0,\cdot) \in \mathrm{Lip}(\Omega, \mathbb{R}^d)$.

$\forall \varepsilon \in (0,1], \quad \forall N \in \mathbb{N}^*,$ consider the solution of the particle system

$$\dot{\xi}^N_{\varepsilon,i}(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_\varepsilon\big(t, x_i^N, x_j^N, \xi^N_{\varepsilon,i}(t)\big)\, \xi^N_{\varepsilon,j}(t), \qquad \xi^N_{\varepsilon,i}(0) = y(0, x_i^N).$$

Then, there exists $C > 0$ such that $\quad \forall N \in \mathbb{N}^* \quad \forall \varepsilon \in (0,1] \quad \forall t \in [0,T]$

$$\Bigg\| \underbrace{\sum_{i=1}^{N} \xi^N_{\varepsilon,i}(t)\, \mathbb{1}_{\Omega_i^N}(\cdot)}_{\text{particles}} - \underbrace{y(t,\cdot)}_{\text{Euler}} \Bigg\|_{L^2(\Omega,\mathbb{R}^d)} \leqslant C\left(\varepsilon + \frac{1}{N^{1/n}} \exp\left(\frac{C}{\varepsilon^{n+p+1}} \exp\left(\frac{C}{\varepsilon^{n+p}}\right)\right)\right).$$

Particle approximations of PDEs

Proof in the linear case:

Assume that $A : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates a C_0 semigroup and that

- $\|e^{tA_\varepsilon}\|_{L(L^2)} \leqslant M\, e^{\beta t} \qquad \forall t \geqslant 0 \qquad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon - A)y\|_{L^2} \lesssim \varepsilon \|y\|_{W^{p+1,\infty}} \qquad \forall y \in W^{p+1,\infty}(\Omega, \mathbb{R}^d) \qquad \forall \varepsilon \in (0,1]$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

First step: convergence of y_ε towards y. By the Duhamel formula:

$$y_\varepsilon(t) - y(t) = \int_0^t e^{(t-\tau)A_\varepsilon}(A_\varepsilon - A)y(\tau)\, d\tau$$

hence

$$\begin{aligned}
\|y_\varepsilon(t) - y(t)\|_{L^2} &\leqslant \int_0^t \left\| e^{(t-\tau)A_\varepsilon}(A_\varepsilon - A)y(\tau) \right\|_{L^2} d\tau \\
&\leqslant \int_0^t M e^{\beta(t-\tau)} \|(A_\varepsilon - A)y(\tau)\|_{L^2}\, d\tau \\
&\lesssim \varepsilon \|y\|_{L^1([0,T],W^{p+1,\infty})} \lesssim \varepsilon \qquad \forall t \in [0,T]
\end{aligned}$$

Particle approximations of PDEs

Proof in the linear case:

Assume that $A : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates a C_0 semigroup and that

- $\|e^{tA_\varepsilon}\|_{L(L^2)} \leqslant M\, e^{\beta t} \qquad \forall t \geqslant 0 \qquad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon - A)y\|_{L^2} \lesssim \varepsilon \|y\|_{W^{p+1,\infty}} \qquad \forall y \in W^{p+1,\infty}(\Omega, \mathbb{R}^d) \qquad \forall \varepsilon \in (0,1]$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

Second step: particle approximation. By Gronwall:

$$\max\left(\|\Xi_\varepsilon^N(t)\|_\infty, \|y_\varepsilon(t)\|_{L^\infty(\Omega,\mathbb{R}^d)}\right) \leqslant e^{t\|\sigma_\varepsilon\|_{L^\infty}} \|y^0\|_{L^\infty(\Omega,\mathbb{R}^d)}$$

By graph limit approximation:

$$\left\| \sum_{i=1}^N \xi_{\varepsilon,i}^N(t)\, \mathbb{1}_{\Omega_i^N}(\cdot) - y_\varepsilon(t,\cdot) \right\|_{L^\infty(\Omega,\mathbb{R}^d)} \leqslant \frac{2C_\Omega}{N^{1/n}} (1 + \mathrm{Lip}(y^0)) e^{2tL_\varepsilon}$$

with $L_\varepsilon = \frac{1}{\varepsilon^{n+p+1}} \exp\left(\frac{1}{\varepsilon^{n+p}}\right)$ (Lipschitz constant on the supports).

Particle approximations of PDEs

Proof in the linear case:

Assume that $A : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates a C_0 semigroup and that

- $\|e^{tA_\varepsilon}\|_{L(L^2)} \leqslant M\, e^{\beta t} \qquad \forall t \geqslant 0 \qquad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon - A)y\|_{L^2} \lesssim \varepsilon \|y\|_{W^{p+1,\infty}} \qquad \forall y \in W^{p+1,\infty}(\Omega, \mathbb{R}^d) \qquad \forall \varepsilon \in (0,1]$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

Conclusion: Up to some constant, by the triangular inequality:

$$\begin{aligned}
\Big\| \sum_{i=1}^{N} \xi^N_{\varepsilon,i}(t)\, \mathbb{1}_{\Omega_i^N}(\cdot) - y(t,\cdot) \Big\|_{L^2} &\leqslant \Big\| \sum_{i=1}^{N} \xi^N_{\varepsilon,i}(t)\, \mathbb{1}_{\Omega_i^N}(\cdot) - y(t,\cdot) \Big\|_{L^\infty} \\
&\leqslant \|y_\varepsilon(t,\cdot) - y(t,\cdot)\|_{L^\infty} + \Big\| \sum_{i=1}^{N} \xi^N_{\varepsilon,i}(t)\, \mathbb{1}_{\Omega_i^N}(\cdot) - y_\varepsilon(t,\cdot) \Big\|_{L^\infty} \\
&\lesssim \varepsilon + \frac{1}{N^{1/n}} \exp\left(\frac{C}{\varepsilon^{n+p+1}} \exp\left(\frac{C}{\varepsilon^{n+p}} \right) \right)
\end{aligned}$$

Particle approximations of PDEs

Proof in the linear case:

Assume that $A : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates a C_0 semigroup and that

- $\|e^{tA_\varepsilon}\|_{L(L^2)} \leqslant M\, e^{\beta t} \qquad \forall t \geqslant 0 \qquad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon - A)y\|_{L^2} \lesssim \varepsilon \|y\|_{W^{p+1,\infty}} \qquad \forall y \in W^{p+1,\infty}(\Omega, \mathbb{R}^d) \qquad \forall \varepsilon \in (0,1]$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

Explicit example of construction of σ_ε: (for $A = \sum a_\alpha D^\alpha$)

Let $\eta \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ nonnegative symmetric s.t. $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$ and let $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$.
Define

$$\sigma_\varepsilon(x,x') = \int_\Omega \eta_\varepsilon(x-z) \sum_{|\alpha|\leqslant p} a_\alpha(z)(D^\alpha \eta_\varepsilon)(z-x')\,dz$$

(double convolution restricted to Ω) so that

$$A_\varepsilon f = \eta_\varepsilon \star_\Omega A(\eta_\varepsilon \star_\Omega f) = \Big(\eta_\varepsilon \star \big(A(\eta_\varepsilon \star (f\,\mathbb{1}_\Omega))\mathbb{1}_\Omega\big)\Big)_{|\Omega}$$

Crucial fact: Like the operator $A - \beta\,\mathrm{id}$, the operator $A_\varepsilon - \beta\,\mathrm{id}$ is m-dissipative on $L^2(\Omega, \mathbb{R}^d)$ because

$$\langle (A_\varepsilon - \beta\,\mathrm{id})f, f\rangle_{L^2(\Omega)} = \langle \eta_\varepsilon \star_\Omega (A-\beta\,\mathrm{id})(\eta_\varepsilon \star_\Omega f), f\rangle_{L^2} = \langle (A-\beta\,\mathrm{id})(\eta_\varepsilon \star_\Omega f), \eta_\varepsilon \star_\Omega f\rangle_{L^2} \leqslant 0$$

Particle approximations of PDEs

Proof in the quasilinear case:

Assume that $\forall z \in L^2(\Omega, \mathbb{R}^d)$, $A(t,z) : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates an evolution system $U(t,s,z)$ and that

- $\|U_\varepsilon(t,s,z)\|_{L(L^2)} \leqslant M\, e^{\beta(t-s)} \qquad \forall t \geqslant s \geqslant 0 \qquad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon(t,z) - A(t,z))y\|_{L^2} \lesssim \varepsilon \|y\|_{W^{p,\infty}} \qquad \forall y \in W^{p+1,\infty}(\Omega, \mathbb{R}^d) \quad \forall \varepsilon \in (0,1]$
- $\|(A_\varepsilon(t,z_1) - A_\varepsilon(t,z_2))y\|_{L^2} \lesssim \|z_1 - z_2\|_{L^2} \|y\|_{W^{p+1,\infty}} \qquad \forall z, z_1, z_2 \in L^2(\Omega, \mathbb{R}^d)$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

Quasilinear theory: Kato 1975 (short version in Pazy, Section 6.4), examples:

- Burgers
- KdV
- quasilinear symmetric hyperbolic systems
- Euler and Navier Stokes (incompressible) in $\mathbb{R}^3$
- coupled Maxwell-Dirac
- quasilinear waves
- magnetohydrodynamics (including compressible fluids)
- etc.

Particle approximations of PDEs

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- $\|(A_\varepsilon(t,z_1) - A_\varepsilon(t,z_2))y\|_{L^2} \lesssim \|z_1 - z_2\|_{L^2} \|y\|_{W^{p+1,\infty}} \qquad \forall z, z_1, z_2 \in L^2(\Omega, \mathbb{R}^d)$
- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

The only difference is in Step 1:

$$\partial_t(y_\varepsilon - y) = A_\varepsilon(y_\varepsilon)y_\varepsilon - A(y)y = A_\varepsilon(y_\varepsilon)(y_\varepsilon - y) + (A_\varepsilon(y_\varepsilon) - A_\varepsilon(y))y + (A_\varepsilon(y) - A(y))y$$

hence (Duhamel)

$$y_\varepsilon(t) - y(t) = \int_0^t U_\varepsilon(t,s,y_\varepsilon(s))\Big((A_\varepsilon(y_\varepsilon(s)) - A_\varepsilon(y(s)))y(s) + (A_\varepsilon(y(s)) - A(y(s)))y(s) \Big) ds$$

thus

$$\|y_\varepsilon(t) - y(t)\|_{L^2} \lesssim \int_0^t \|y_\varepsilon(s) - y(s)\|_{L^2}\, ds + \varepsilon \quad \overset{\text{Gronwall}}{\Longrightarrow} \quad \|y_\varepsilon(t) - y(t)\|_{L^2} \lesssim \varepsilon$$

Particle approximations of PDEs

Proof in the quasilinear case:

Assume that $\forall z \in L^2(\Omega, \mathbb{R}^d)$, $A(t,z) : D(A) \to L^2(\Omega, \mathbb{R}^d)$ generates an evolution system $U(t,s,z)$ and that

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- $\|\sigma_\varepsilon\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{n+p}}$ and $\mathrm{Lip}(\sigma_\varepsilon) \lesssim \frac{1}{\varepsilon^{n+p+1}}$

Explicit example of construction of σ_ε: (for $A(t,\xi) = \sum a_\alpha(t,x,\xi) D^\alpha$)

$$\sigma_\varepsilon(t,x,x',\xi) = \int_\Omega \eta_\varepsilon(x-z) \sum_{|\alpha| \leqslant p} a_\alpha(t,z,\xi)(D^\alpha \eta_\varepsilon)(z-x')\,dz$$

(double convolution restricted to Ω) so that $A_\varepsilon(t,\xi) f = \eta_\varepsilon \star_\Omega A(t,\xi)(\eta_\varepsilon \star_\Omega f)$

All in all, we have obtained

$$\Big\| \underbrace{\sum_{i=1}^N \xi^N_{\varepsilon,i}(t)\, \mathbb{1}_{\Omega^N_i}(\cdot)}_{\textit{particles}} \; - \; \underbrace{y(t,\cdot)}_{\textit{Euler}} \Big\|_{L^2(\Omega,\mathbb{R}^d)} \leqslant C\left(\varepsilon + \frac{1}{N^{1/n}} \exp\left(\frac{C}{\varepsilon^{n+p+1}} \exp\left(\frac{C}{\varepsilon^{n+p}}\right)\right)\right).$$

Particle approximations of PDEs

To take limits $N \to +\infty$ and $\varepsilon \to 0$, we must choose $\frac{1}{N^{1/n}} \exp\left(\frac{C}{\varepsilon^{n+p+1}} \exp\left(\frac{C}{\varepsilon^{n+p}}\right)\right) \to 0$.

Optimizing leads to

$$\varepsilon_N \sim \left(\frac{C}{\ln \ln N}\right)^{\frac{1}{n+p}}$$

and then

$$\boxed{\Big\| \underbrace{\sum_{i=1}^{N} \xi_{\varepsilon,i}^N(t)\, \mathbb{1}_{\Omega_i^N}(\cdot)}_{\textit{particles}} - \underbrace{y(t,\cdot)}_{\textit{Euler}} \Big\|_{L^2(\Omega,\mathbf{R}^d)} \leqslant \left(\frac{C}{\ln \ln N}\right)^{\frac{1}{n+p}}}$$

Similar estimates have been obtained by:

– Bodineau Gallagher Saint-Raymond (linear Boltzmann to heat by hydrodynamic limit)

– Slepcev (Leçons Jacques-Louis Lions, 2021) for heat-like equations.

Here, we have a particle approximation for *arbitrary (well-posed) quasilinear PDEs*.

What does this result mean?

In statistical physics: Ω of volume 1 contains

$$N \simeq 6.10^{23}$$

particles (Avogadro number). But $\ln \ln N \simeq 4$!! Note that

$$\log_{10} \log_{10} 10^{10} = 1...$$

Actually $\frac{1}{\ln \ln N}$ is a kind of physical barrier.

Perspectives

- Understand what the latter result implies.
- Investigate under which (physical?) assumptions the estimates can be improved, and investigate numerical consequences.
- Investigate more general nonlinear PDEs.
- How to close the hierarchy of equations for coupled moments? (BBGKY-like hierarchy) Maybe, introduce a small parameter ε.
- Consider particle dynamics with "triplewise" interactions:

$$\dot{\xi}_i(t) = \frac{1}{N^2} \sum_{j,k=1}^{N} G(t, x_i, x_j, x_k, \xi_i(t), \xi_j(t), \xi_k(t))$$

  and their various limits.
- Add some controls to all equations, and show how to perform the various passages to the limit, also in the control strategies.