Interplay between numerical methods and evolution PDEs on graphs

Joint works with G. Heinze (WIAS Berlin), L. Mikolás (Oxford), F. S. Patacchini (IFPEN), A. Schlichting (WWU Münster), and D. Slepčev (CMU Pittsburgh)

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Workshop "Numerical methods for optimal transport problems, mean field games, and multi-agent dynamics"
Universidad Técnica Federico Santa María
Valparaíso, January 8-12, 2024

Oxford Mathematics

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 883363)

European Research Council
Established by the European Commission

Motivation: why evolution PDEs on graphs?

- Social networks: polarisation and formation of echo chambers

  F. Baumann, P. Lorenz-Spreen, I. M. Sokolov, M. Starnini., *Phys. Rev. Lett*, 2020

  A. Benatti, H. F. de Arruda, F. N. Silva, C. H. Comin, L. da Fontoura Costa, *Journal of Statistical Mechanics: Theory and Experiment*, 2020

- Transportation Newtorks: gravity interactions

  K. Tamura, H. Takayasu, M. Takayasu, *Scientific Reports*, 2018

  H. Koike, H. Takayasu, and M. Takayasu, *Journal of Statistical Physics*, 2022

- Data Science/Machine Learning: data representation as point clouds for clustering and classification

  M. Belkin, P. Niyogi, *Neural Comput.*, 2002

  R. R. Coifman, S. Lafon, *Appl. Comput. Harmon. Anal.*, 2006

  K. Craig, N. Garcia-Trillos, N. Garcia, D. Slepcev, *Springer International Publishing*, 2022.

Outline

Dynamics on graphs: well-posedness, gradient flow structure, and graph limit

Localising the graph

Co-evolving graphs

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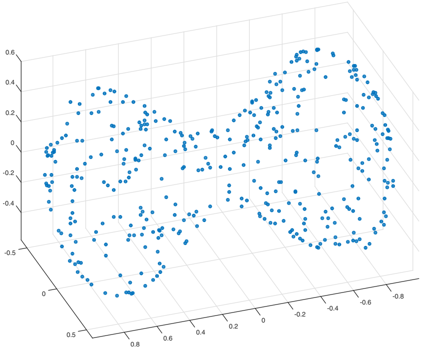
Localising the graph

Co-evolving graphs

Notation

- $X = \{x_1, x_2, ..., x_n\}$ random sample i.i.d. according to $\mu \in \mathcal{M}^+(\mathbb{R}^d)$
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$



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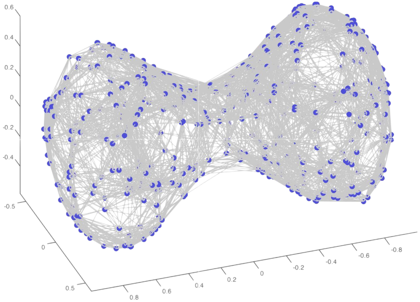
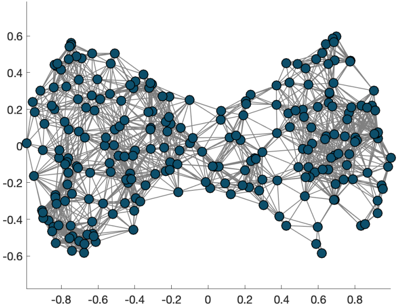
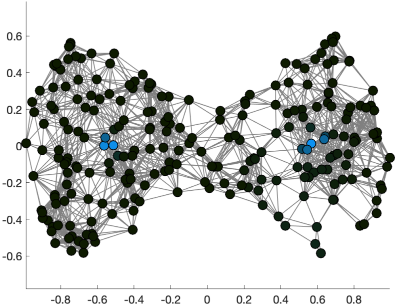
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
- a symmetric weight function $\eta : D \to [0, \infty)$ with
  $D := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{x = y\}$
  $\Rightarrow$ (μ^n, η) defines an undirected discrete weighted graph



Example: dynamics driven by interaction energies on graphs





Video

Example: dynamics driven by interaction energies on graphs

$$\mathcal{E}_X(\rho) = \frac{1}{2}\sum_{x\in X}\sum_{y\in X} K_{x,y}\rho_x\rho_y \tag{1}$$

On $\mathbb{R}^d$:

$$\dot{x}_i = -\sum_{j=1}^{n} \rho_j \nabla_x K(x_i, x_j) \tag{2}$$

On finite graphs

$$\frac{d\rho_x}{dt} = -\sum_{y\in X} j_{x,y}\eta(x,y) \tag{3}$$

$$j_{x,y} = \phi\big(\rho_x, \rho_y\big)\, v_{x,y} \tag{4}$$

$$v_{x,y} = -\sum_{z\in X} \rho_z (K_{y,z} - K_{x,z}). \tag{5}$$

CHOICE IS NOT CANONICAL!

Goals

- Define gradient flow of interaction energy on graph (μ, η)
- Dynamics stable under graph limit $n \to \infty$ (discrete-to-continuum)
- Dynamics stable for local limit: $\mu = \mathrm{Leb}(\mathbb{R}^d)$, $\eta^\varepsilon(x,y) = \varepsilon^{-d-2}\eta\left(\frac{x-y}{\varepsilon}\right)$
  $\Rightarrow$ limit $\varepsilon \to 0$ should give $\partial_t \rho = \nabla\cdot(\rho\nabla K * \rho)$

Example: dynamics driven by interaction energies on graphs

General framework

- $\mathbb{R}^d$ set of possible vertices, $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ set of possible edges
- $\eta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0, \infty)$ symmetric weight function
- $G := \{\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} | \eta(x, y) > 0\}$ set of edges
- $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ set of vertices
- $\rho \in \mathcal{P}(\mathbb{R}^d)$ distribution of mass

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Evolution of interest

Gradient descent of the energy $\mathcal{E} : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathcal{E}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y) \, d\rho(x) \, d\rho(y),$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is symmetric.

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Continuum (local) setting: NLIE

$\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$ is a Wasserstein gradient flow for $\mathcal{E}$[a]

[a] J.A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent, D. Slepčev - Duke Math. J. 156 (2011)

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What is the analogue of the NLIE on a graph?

Nonlocal continuity equation

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Continuity equation

$$\partial_t \rho_t + \nabla \cdot j_t = 0 \qquad \text{where} \qquad j_t(x) := \rho_t(x) v_t(x)$$

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On Graphs

$$\partial_t \rho_t(x) + (\overline{\nabla} \cdot j_t)(x) = \partial_t \rho_t(x) + \int_{\mathbb{R}^d} j_t(x,y)\, \eta(x,y)\, dy = 0$$

$$j_t(x,y) = \phi\big(\rho_t(x), \rho_t(y)\big)\, v_t(x,y)$$

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Upwind interpolation: density along edges = density at the source

$$j_t(x, y) = \rho(x) v_t(x, y)_+ - \rho(y) v_t(x, y)_-$$

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Continuity equation

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Nonlocal continuity equation $(\rho_t \ll \mu)$

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \big(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\big)\, \eta(x,y)\, d\mu(y) = 0 \qquad \text{(NCE)}$$

Nonlocal interaction equation on graphs: NL^2IE

(NCE) with $v_t^{\mathcal{E}} := -\overline{\nabla} \frac{\delta \mathcal{E}}{\delta \rho} = -\overline{\nabla} K * \rho_t$

$$\partial_t \rho + \overline{\nabla} \cdot F^{\Phi}[\mu; \rho_t, v_t] = 0 \qquad \text{(NCE)}$$

Definition (Admissible flux interpolation)

A measurable function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ is called an admissible flux interpolation provided that the following conditions hold:

(i) Φ satisfies

$$\Phi(0,0;v) = \Phi(a,b;0) = 0, \qquad \text{for all } a,b,v \in \mathbb{R}; \tag{1}$$

(ii) Φ is argument-wise Lipschitz in the sense that, for some $L_\Phi > 0$, any $a,b,c,d,v,w \in \mathbb{R}$, it holds

$$|\Phi(a,b;w) - \Phi(a,b;v)| \le L_\Phi(|a| + |b|)|w - v|; \tag{2a}$$

$$|\Phi(a,b;v) - \Phi(c,d;v)| \le L_\Phi(|a - c| + |b - d|)|v|; \tag{2b}$$

(iii) Φ is positively one-homogeneous in its first and second argument, that is, for all $\alpha > 0$ and $(a,b,w) \in \mathbb{R}^3$, it holds

$$\Phi(\alpha a, \alpha b; w) = \alpha \Phi(a,b;w).$$

General interpolations: examples

- Upwind interpolation. One important case is given by the upwind interpolation Φ_{upwind} defined as

$$\Phi_{\mathrm{upwind}}(a, b; w) = aw_+ - bw_- \qquad \text{for } (a, b, w) \in \mathbb{R}^3. \tag{3}$$

- Mean multipliers. Another case is product interpolation Φ_{prod}, which is of the form

$$\Phi_{\mathrm{prod}}(a, b; w) = \phi(a, b)w \qquad \text{for } (a, b, w) \in \mathbb{R}^3,$$

  with $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ any measurable function satisfying, for some $L_\Phi > 0$,

$$\begin{aligned} &|\phi(a, b)| \le L_\Phi \max\{|a|, |b|\}, \\ &|\phi(a, b) - \phi(c, d)| \le L_\Phi(|a - c| + |b - d|), \\ &\phi(\alpha a, \alpha b) = \alpha\phi(a, b), \\ &\phi(a, b) = \phi(b, a), \end{aligned}$$

  for all $\alpha \ge 0$ and $a, b, c, d \in \mathbb{R}$. Common choices for ϕ are as below:
    - *Arithmetic mean.* $\phi_{\mathrm{AM}}(a, b) := \frac{a+b}{2}$;
    - *Maximal mean.* $\phi_{\max}(a, b) := \max\{a, b\}$.

Definition (Admissible flux)

Let Φ be an admissible flux interpolation, and let $\rho \in \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d)$ and $w \in \mathcal{V}^{\mathrm{as}}(G) := \{v\colon G \to \mathbb{R} : v(x,y) = -v(y,x)\}$. Furthermore, take $\lambda \in \mathcal{M}^+(\mathbb{R}^{2d})$ such that $\rho \otimes \mu, \mu \otimes \rho \ll \lambda$ (e.g., $\lambda = |\rho| \otimes \mu + \mu \otimes |\rho|$). Then, the admissible flux $F^\Phi[\mu;\rho,w] \in \mathcal{M}(G)$ at (ρ, w) is defined by

$$\mathrm{d}F^\Phi[\mu;\rho,w] = \Phi\left(\frac{\mathrm{d}(\rho\otimes\mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu\otimes\rho)}{\mathrm{d}\lambda}; w\right)\mathrm{d}\lambda. \tag{4}$$

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Definition (Measure-valued solution to the NCL)

Given Φ and a measurable $V \colon [0,T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \to \mathcal{V}^{\mathrm{as}}(G)$, a curve $\rho \colon [0,T] \to \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d)$ is a measure-valued solution to the NCL, denoted as

$$\partial_t \rho + \overline{\nabla} \cdot F^\Phi[\mu; \rho, V_t(\rho)] = 0, \tag{NCL}$$

provided that, for any $A \in \mathcal{B}(\mathbb{R}^d)$, it holds that

(i) $\rho \in \mathcal{AC}_t$;

(ii) $t \mapsto \overline{\nabla} \cdot F^\Phi[\mu; \rho_t, V_t(\rho_t)][A] \in L^1([0,T])$;

(iii) ρ satisfies

$$\rho_t[A] + \int_0^t \overline{\nabla} \cdot F^\Phi[\mu; \rho_s, V_s(\rho_s)][A] \mathrm{d}s = \rho_0[A] \qquad \text{for a.e. } t \in [0,T]. \tag{5}$$

Theorem (A. E., F. S. Patacchini, A. Schlichting, EJAM '23)

Let $V\colon [0,T] \times \mathcal{M}^{M}_{\mathrm{TV}}(\mathbb{R}^d) \to \mathcal{V}^{\mathrm{as}}(G)$ *and suppose there are constants* $C_V, L_V > 0$ *so that, for all* $t \in [0,T]$ *and all* $\rho, \sigma \in \mathcal{M}^{M}_{\mathrm{TV}}(\mathbb{R}^d)$*,*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{x\}} |V_t[\rho](x,y)|\eta(x,y)d\mu(y) \le C_V,$$

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{x\}} |V_t[\rho](x,y) - V_t[\sigma](x,y)|\eta(x,y)d\mu(y) \le L_V\|\rho-\sigma\|_{TV}.$$

Then, there exists a unique measure solution ρ *to* (NCL) *such that* $\rho_0 = \rho^0$*.*

Proof via Banach Fixed-Point Theorem

Corollary (Well-posedness for NL^{2}IE)

Assume that η *satisfies*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y)\eta(x,y)d\mu(y) < \infty \tag{6}$$

for some nonnegative measurable function $f\colon \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$*. Let* $K\colon \mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ *and* $P\colon \mathbb{R}^d \to \mathbb{R}$ *be such that there exist constants* $L_K, L_P > 0$ *for which*

$$|K(y,z) - K(x,z)| \le L_K f(x,y), \quad |P(y) - P(x)| \le L_P f(x,y), \tag{7}$$

for all $x,y,z \in \mathbb{R}^d$*. Then, NL*^{2}*IE has a unique measure solution* ρ *such that* $\rho_0 = \rho^0$*.*

Wasserstein-like gradient flow structure

- [Maas '11] / [Mielke '11] / [Chow, Huang, Li, Zhou '12]
  Diffusion on graphs as gradient flows of the entropy
  ⇒ Wassertein metric on a finite graph

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Gradient flows for free energies/(relative) entropies:

$$\mathcal{F}^\sigma(\rho) = \sigma \int \rho(x) \log \rho(x)\, dx + \frac{1}{2} \iint K(x,y)\, d\rho(x)\, d\rho(y)$$

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What if $\sigma = 0$?

$\sigma \to 0$: nonlocal metrics above do not have a clear/well-defined limit!

What is a suitable metric for gradient structure of interaction energies?

Upwind transportation "metric"

Upwind transportation “metric”

Nonlocal continuity equation ($\rho_t \ll \mu$)

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \left(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\right) \eta(x,y)\, d\mu(y) = 0 \tag{NCE}$$

Benamou-Brenier

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \rho_t(x)\, dx\, dt \;\middle|\; (\rho_t, v_t) \in \mathrm{CE}(\rho_0, \rho_1) \right\}$$

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Upwind nonlocal transportation "metric": Benamou-Brenier

$$\inf_{(\rho,v)\in \mathrm{NCE}} \left\{ \frac{1}{2} \int_0^1 \iint_G \left(|v_t(x,y)_+|^2 \rho_t(x) + |v_t(x,y)_-|^2 \rho_t(y)\right) \eta(x,y)\, d\mu(x)\, d\mu(y)\, dt \right\}$$

Note that:

- ρ might contain atoms, even if μ is Lebesgue!
  $\Rightarrow$ measure valued framework
- Benamou-Brenier functional is not jointly convex in (ρ_t, v_t)
  $\Rightarrow$ flux variables

Definition

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\rho \in \mathcal{P}(\mathbb{R}^d)$ and $\boldsymbol{j} \in \mathcal{M}(G)$, consider $\lambda \in \mathcal{M}(G)$ such that $\rho \otimes \mu, \mu \otimes \rho, |\boldsymbol{j}| \ll |\lambda|$. We define

$$\mathcal{A}(\mu; \rho, \boldsymbol{j}) = \frac{1}{2} \iint_G \left(\alpha \left(\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\rho \otimes \mu)}{d|\lambda|} \right) + \alpha \left(-\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\mu \otimes \rho)}{d|\lambda|} \right) \right) \eta \, d|\lambda|. \qquad (8)$$

Hereby, the lower semicontinuous, convex, and positively one-homogeneous function $\alpha \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is defined, for all $j \in \mathbb{R}$ and $r \geq 0$, by

$$\alpha(j, r) := \begin{cases} \frac{(j_+)^2}{r} & \text{if } r > 0, \\ 0 & \text{if } j \leq 0 \text{ and } r = 0, \\ \infty & \text{if } j > 0 \text{ and } r = 0, \end{cases} \qquad (9)$$

with $j_+ = \max\{0, j\}$. If the measure μ is clear from the context, we write $\mathcal{A}(\rho, \boldsymbol{j})$ for $\mathcal{A}(\mu; \rho, \boldsymbol{j})$.

Definition

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\rho \in \mathcal{P}(\mathbb{R}^d)$ and $\boldsymbol{j} \in \mathcal{M}(G)$, consider $\lambda \in \mathcal{M}(G)$ such that $\rho \otimes \mu, \mu \otimes \rho, |\boldsymbol{j}| \ll |\lambda|$. We define

$$\mathcal{A}(\mu; \rho, \boldsymbol{j}) = \frac{1}{2} \iint_G \left(\alpha \left(\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\rho \otimes \mu)}{d|\lambda|} \right) + \alpha \left(-\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\mu \otimes \rho)}{d|\lambda|} \right) \right) \eta \, d|\lambda|. \tag{8}$$

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If $\rho \ll \mu$ and $\boldsymbol{j} \ll \mu \otimes \mu$

$$\mathcal{A}(\mu; \rho, \boldsymbol{j}) = \frac{1}{2} \iint_G \left(\frac{(j(x,y)_+)^2}{\rho(x)} + \frac{(j(x,y)_-)^2}{\rho(y)} \right) \eta(x,y) \, d\mu(x) \, d\mu(y)$$

Nonlocal upwind transportation quasi-metric

Definition

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying moment bound and local blow-up control, and $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, the nonlocal upwind transportation cost between ρ_0 and ρ_1 is defined by

$$\mathcal{T}_\mu(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t)\, dt : (\rho, \boldsymbol{j}) \in \mathrm{NCE}(\rho_0, \rho_1) \right\}. \tag{10}$$

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Properties

- The infimum is attained for $(\rho, \boldsymbol{j}) \in \mathrm{NCE}(\rho_0, \rho_1)$ with $\mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t) = \mathcal{T}_\mu(\rho_0, \rho_1)^2$

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- $\{\rho_t\}_{t\in[0,T]} \in \mathrm{AC}([0,T]; (\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\mu))$ iff $\exists\, (\boldsymbol{j}_t)_{t\in[0,T]}$ such that $(\rho, \boldsymbol{j}) \in \mathrm{CE}_T$ and $\int_0^T \sqrt{\mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t)}\, dt < \infty$

D. Slepčev, A. Warren, *Nonlocal wasserstein distance: metric and asymptotic properties* - CVPDE '23

Gradient descent in Finsler geometry [Ohta-Sturm '09, '12] - [Agueh '12]

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Inner product

$\boldsymbol{j} \in T_\rho \mathcal{P}_2(\mathbb{R}^d)$, we define an inner product $g_{\rho,\boldsymbol{j}} \colon T_\rho \mathcal{P}_2(\mathbb{R}^d) \times T_\rho \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

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Nonlocal interaction energy

$$\mathrm{grad}^- \mathcal{E}(\rho)(x,y) = -\overline{\nabla}(K*\rho)(x,y) \left(\rho(x) \chi_{\{-\overline{\nabla}K*\rho>0\}}(x,y) + \rho(y) \chi_{\{-\overline{\nabla}K*\rho<0\}}(x,y) \right)$$

Theorem

A curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$ is a weak solution to (NL^{2}IE) if and only if ρ belongs to $\mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$ and is a curve of maximal slope for $\mathcal{E}$ with respect to $\sqrt{\mathcal{D}}$, that is, satisfies

$$\mathcal{G}_T(\rho) = 0.$$

Local slope & De Giorgi Functional

For any $\rho \in \mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$, the De Giorgi functional at ρ is defined as

$$\mathcal{G}_T(\rho) := \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}(\rho_\tau) + |\rho'_\tau|^2\right) d\tau \geq 0,$$

Variational characterisation of (NL^{2}IE)

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$$\mathcal{D}(\rho) := \widehat{g}_{\rho,-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right)$$

$$= -\iint_G \left|\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(x,y)_-\right|^2 \eta(x,y)\, d\rho(x)\, d\mu(y)$$

Stability of gradient flows

Let $(\mu^n)_n \subset \mathcal{M}^+(\mathbb{R}^d)$ and suppose that $(\mu^n)_n$ narrowly converges to μ. Suppose that ρ^n is a gradient flow of $\mathcal{E}$ with respect to μ^n for all $n \in \mathbb{N}$, that is,

$$\mathcal{G}_T(\mu^n; \rho^n) = 0 \quad \text{for all } n \in \mathbb{N},$$

such that $(\rho_0^n)_n$ satisfies $\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < \infty$ and $\rho_t^n \rightharpoonup \rho_t$ as $n \to \infty$ for all $t \in [0, T]$ for some curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$. Then, $\rho \in AC([0, T]; (\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\mu))$ and ρ is a gradient flow of $\mathcal{E}$ with respect to μ, that is,

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Stability with respect to graph approximations

Stability of gradient flows

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$$\mathcal{G}_T(\mu; \rho) = 0.$$

Corollary

Existence of weak solution to (NL^{2}IE) via finite-dimensional approximation.

A. E., F. S. Patacchini, A. Schlichting, D. Slepčev, *Nonlocal-interaction equation on graphs: gradient flow structure and continuum limit* - ARMA (2021).

Outline

Dynamics on graphs: well-posedness, gradient flow structure, and graph limit

Localising the graph

Co-evolving graphs

Graph-to-local limit

Consider a localising graph (μ, η^ε), for

$$\eta^\varepsilon(x,y) := \frac{1}{\varepsilon^{d+2}}\vartheta\left(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\right) \qquad (\eta)$$

$$\partial_t \rho_t^\varepsilon(x) + \int_{\mathbb{R}^d} \overline{\nabla}(K*\rho_t^\varepsilon)(x,y)_- \eta^\varepsilon(x,y)\rho_t^\varepsilon(x)\mathrm{d}\mu(y) - \int_{\mathbb{R}^d} \overline{\nabla}(K*\rho_t^\varepsilon)(x,y)_+ \eta^\varepsilon(x,y)\mathrm{d}\rho_t^\varepsilon(y) = 0 \qquad (\mathrm{NL}^2\mathrm{IE}_\varepsilon)$$

$$\Big\downarrow \varepsilon\to 0$$

$$\partial_t \rho_t = \mathrm{div}(\rho_t \mathbb{T}(\nabla K * \rho_t)) \qquad (\mathrm{NLIE}_\mathbb{T})$$

The tensor $\mathbb{T}:\mathbb{R}^d \to \mathbb{R}^{d\times d}$ is of the form

$$\mathbb{T}(x) := \frac{1}{2}\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d}(x)\int_{\mathbb{R}^d\setminus\{0\}} w\otimes w\,\vartheta(x,w)\mathrm{d}w. \qquad (\mathbb{T})$$

- S. Lisini - ESAIM Control Optim. Calc. Var. (2009) diffusion
- D. Forkert, J. Maas, and L. Portinale - SIMA (2022) Evolutionary Γ-convergence for FP
- A. Hraivoronska, O.Tse - SIMA (2023) limiting behaviour of random walks on tessellations

Proposition (Local flux)

Let $j \in \mathcal{M}(\mathbb{R}^{2d}_{\not\Delta})$ satisfy the integrability condition $\iint_{\mathbb{R}^{2d}_{\not\Delta}} |x-y|\eta(x,y)|j|(x,y) < \infty$. Then there exists $\hat{\jmath} \in \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$\frac{1}{2}\iint_{\mathbb{R}^{2d}_{\not\Delta}} \overline{\nabla}\varphi\,\eta dj = \int_{\mathbb{R}^d} \nabla\varphi \cdot d\hat{\jmath}, \qquad \text{for all } \varphi \in C_c^1(\mathbb{R}^d). \tag{11}$$

In particular, if $(\boldsymbol{\rho},\boldsymbol{j}) \in \mathsf{NCE}_T$ such that $\mathcal{A}(\mu,\eta;\boldsymbol{\rho},\boldsymbol{j}) < \infty$, then there exists $(\hat{\jmath}_t)_{t\in[0,T]} \subset \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that $(\boldsymbol{\rho},\hat{\boldsymbol{\jmath}}) \in \mathsf{CE}_T$.

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Idea of the proof.

$$\begin{aligned}\varphi(y)-\varphi(x) &= \int_0^{|y-x|} \nabla\varphi(x+s\nu_{x,y})\cdot\nu_{x,y}\mathrm{d}s = \int_{[[x,y]]} \nabla\varphi(\xi)\cdot\nu_{x,y}\mathrm{d}\mathcal{H}^1(\xi)\\ &= \int_{\mathbb{R}^d} \nabla\varphi(\xi)\cdot\nu_{x,y}\mathrm{d}\sigma_{x,y}(\xi).\end{aligned} \tag{12}$$

$$\sigma_{x,y}[A] = \mathcal{H}^1(A\cap[[x,y]]) \quad \text{with} \quad [[x,y]] := \Big\{(1-s)x+sy \in \mathbb{R}^d : s\in[0,1]\Big\}.$$

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In particular, if $(\boldsymbol{\rho},\boldsymbol{j}) \in \mathsf{NCE}_T$ such that $\mathcal{A}(\mu,\eta;\boldsymbol{\rho},\boldsymbol{j}) < \infty$, then there exists $(\hat{\jmath}_t)_{t\in[0,T]} \subset \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that $(\boldsymbol{\rho},\hat{\boldsymbol{\jmath}}) \in \mathsf{CE}_T$.

Proposition (Compactness)

Let $(\mu^\varepsilon)_{\varepsilon>0} \subset \mathcal{M}^+(\mathbb{R}^d)$ and $(\eta^\varepsilon)_{\varepsilon>0}$ identify localising graphs, uniformly in ε. Let $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{j}^\varepsilon)_{\varepsilon>0} \subset \mathsf{NCE}_T$ be such that $\sup_{\varepsilon>0}\mathcal{A}(\mu^\varepsilon,\eta^\varepsilon;\boldsymbol{\rho}^\varepsilon,\boldsymbol{j}^\varepsilon) < \infty$ and let $\hat{\boldsymbol{\jmath}}^\varepsilon$ be associated to $\boldsymbol{j}^\varepsilon$ as in Proposition above. Then there exists a (not relabeled) subsequence of pairs $(\boldsymbol{\rho}^\varepsilon,\hat{\boldsymbol{\jmath}}^\varepsilon) \in \mathsf{CE}_T$ and a pair $(\boldsymbol{\rho},\hat{\boldsymbol{\jmath}}) \in \mathsf{CE}_T$ such that $\rho_t^\varepsilon \rightharpoonup \rho_t$ narrowly in $\mathcal{P}(\mathbb{R}^d)$ for a.e. $t \in [0,T]$ and such that $\int_\cdot \hat{\jmath}_t^\varepsilon dt \overset{*}{\rightharpoonup} \int_\cdot \hat{\jmath} dt$ weakly-$*$ in $\mathcal{M}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$.

Limiting tensor structure

Space of tangent velocities

$$\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) := \left\{ v : G^{\varepsilon} \to \mathbb{R} : v_+ \mathrm{d}(\rho \otimes \mu) - v_- \mathrm{d}(\mu \otimes \rho) \in T^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) \right\} \quad (12)$$

$\{\overline{\nabla}\varphi : \varphi \in C_c^{\infty}(\mathbb{R}^d)\}$ is dense in $\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)$ wrt "L^2-norm"

Tangent-to-cotangent mapping

$\widetilde{I}^{\varepsilon}_{\rho} : \widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) \to \left(\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)\right)^*$, for a fixed $v \in \widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)$

$$\widetilde{I}^{\varepsilon}_{\rho}(v)[w] := \frac{1}{2} \iint_{G^{\varepsilon}} w\eta^{\varepsilon}[v_+ \mathrm{d}(\rho \otimes \mu) - v_- \mathrm{d}(\mu \otimes \rho)] \quad (13)$$

$$\begin{aligned} \widetilde{I}^{\varepsilon}_{\rho}(\overline{\nabla}\varphi)[\overline{\nabla}\psi] &= \iint_{G^{\varepsilon}} (\overline{\nabla}\varphi)_+(x,y)\overline{\nabla}\psi(x,y)\eta^{\varepsilon}(x,y)\mathrm{d}\rho^{\varepsilon}(x)\mathrm{d}\mu(y) \\ &= \frac{1}{2}\iint_{G^{\varepsilon}} \overline{\nabla}\varphi(x,y)\overline{\nabla}\psi(x,y)\eta^{\varepsilon}(x,y)\mathrm{d}\rho(x)\mathrm{d}\mu(y) + o(1) \\ &= \int_{\mathbb{R}^d} \nabla\varphi(x)\cdot\mathbb{T}^{\varepsilon}(x)\nabla\psi(x)\mathrm{d}\rho(x) + o(1) \end{aligned}$$

$$\mathbb{T}^{\varepsilon}(x) := \frac{1}{2}\int_{\mathbb{R}^d\setminus\{x\}} (x-y)\otimes(x-y)\,\eta^{\varepsilon}(x,y)\mathrm{d}\mu(y).$$

Theorem (Limiting inner product)

The tangent-to-cotangent mapping $\widetilde{I}^\varepsilon_\rho : \widetilde{T}^\varepsilon_\rho \mathcal{P}_2(\mathbb{R}^d) \to \left(\widetilde{T}^\varepsilon_\rho \mathcal{P}_2(\mathbb{R}^d)\right)^*$ *defined in* (13) *satisfies*

$$\lim_{\varepsilon\to 0} \widetilde{I}^\varepsilon_{\rho^\varepsilon}(\overline{\nabla}\varphi)[\overline{\nabla}\psi] = \int_{\mathbb{R}^d} \nabla\varphi \cdot \mathbb{T}\nabla\psi d\rho, \qquad \forall \varphi, \psi \in C^2_c(\mathbb{R}^d),$$

with the tensor $\mathbb{T} \in C(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *obtained as limit of* $(\mathbb{T}^\varepsilon)_{\varepsilon_0 \geq \varepsilon > 0}$. *The limiting tensor, given by*

$$\mathbb{T}(x) := \frac{1}{2}\widetilde{\mu}(x) \int_{\mathbb{R}^d\setminus\{0\}} w \otimes w \, \vartheta(x, w) dw, \tag{$\mathbb{T}$}$$

is bounded and uniformly continuous.
Furthermore, the tensor $\mathbb{T}$ *is uniformly elliptic, i.e. there exist* $c, C > 0$ *such that for any* $x, \xi \in \mathbb{R}^d$ *we have*

$$c|\xi|^2 \leq \xi \cdot \mathbb{T}(x)\xi \leq C|\xi|^2.$$

Finally, for any $x \in \mathbb{R}$ *the matrix* $\mathbb{T}(x)$ *is symmetric.*

Variational graph-to-local limit

Theorem (Graph-to-local limit)

Let (μ, η^ε) be a localising graph. For any $\varepsilon > 0$ suppose that $\boldsymbol{\rho}^\varepsilon$ is a gradient flow of $\mathcal{E}$ in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\varepsilon))$, that is,

$$\mathcal{E}(\rho_T^\varepsilon) - \mathcal{E}(\rho_0^\varepsilon) + \frac{1}{2}\int_0^T \left(\mathcal{D}_\varepsilon(\rho_\tau^\varepsilon) + |\rho_\tau'|_\varepsilon^2\right) d\tau = 0 \quad \text{for any } \varepsilon > 0,$$

with $(\rho_0^\varepsilon)_\varepsilon \subset \mathcal{P}_2(\mathbb{R}^d)$ be such that $\sup_{\varepsilon>0} M_2(\rho_0^\varepsilon) < \infty$. Then there exists $\boldsymbol{\rho} \in \mathrm{AC}^2([0,T]; (\mathcal{P}_2(\mathbb{R}^d_{\mathbb{T}}), W_{\mathbb{T}}))$ such that $\rho_t^\varepsilon \rightharpoonup \rho_t$ as $\varepsilon \to 0$ for all $t \in [0,T]$ and $\boldsymbol{\rho}$ is a gradient flow of $\mathcal{E}$ in $(\mathcal{P}_2(\mathbb{R}^d_{\mathbb{T}}), W_{\mathbb{T}}))$, that is,

$$\mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}_{\mathbb{T}}(\rho_\tau) + |\rho_\tau'|_{\mathbb{T}}^2\right) d\tau = 0,$$

where the metric slope is

$$\mathcal{D}_{\mathbb{T}}(\rho) = \int_{\mathbb{R}^d} \left\langle \nabla \frac{\delta \mathcal{E}}{\delta \rho}, \mathbb{T}\nabla \frac{\delta \mathcal{E}}{\delta \rho} \right\rangle d\rho.$$

$$W_{\mathbb{T}}^2(\varrho_0, \varrho_1) = \inf\left\{ \int_0^1 \int_{\mathbb{R}^d} \left\langle \mathbb{T}^{-1}(x) \frac{\mathrm{d}j}{\mathrm{d}\rho}(x), \frac{\mathrm{d}j}{\mathrm{d}\rho}(x) \right\rangle \mathrm{d}\rho(x)\mathrm{d}t : (\boldsymbol{\rho}, \boldsymbol{j}) \in \mathrm{CE}(\varrho_0, \varrho_1) \right\}$$

A. E., G. Heinze, A. Schlichting, *Graph-to-local limit for the nonlocal interaction equation*, preprint arXiv:2306.03475.

Outline

Dynamics on graphs: well-posedness, gradient flow structure, and graph limit

Localising the graph

Co-evolving graphs

Co-evolving graphs

$$\begin{aligned} \partial_t \rho_t &= -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]], \\ \partial_t \eta_t &= \omega[\rho_t] - \eta_t, \end{aligned} \tag{Co-NCL}$$

$$\mathrm{d}F^{\Phi}[\mu, \eta; \rho, w] = \Phi\left(\frac{\mathrm{d}(\rho \otimes \mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu \otimes \rho)}{\mathrm{d}\lambda}; w\right) \eta \, \mathrm{d}\lambda.$$

$$\begin{aligned} \partial_t \rho_t &= -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]], \\ \partial_t \eta_t &= \omega[\rho_t] - \eta_t, \end{aligned} \tag{Co-NCL}$$

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Definition (Solution to (Co-NCL))

Given an admissible Φ, a $V : [0, T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times \mathbb{R}^{2d}_{\not\triangle} \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\not\triangle})$, and function $\omega : [0, T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times \mathbb{R}^{2d}_{\not\triangle} \to \mathbb{R}$, a pair $(\rho, \eta) : [0, T] \to \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times C_b(\mathbb{R}^{2d}_{\not\triangle})$ is a solution to the initial value problem (Co-NCL) if, for any $\varphi \in C_0(\mathbb{R}^d)$,

1. $\rho \in AC([0, T], \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d))$, $\eta \in AC([0, T], C_b(\mathbb{R}^{2d}_{\not\triangle}))$;
2. the maps $t \mapsto \langle \varphi, \overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \rangle$ and $t \mapsto \omega[\rho_t] - \eta_t \in L^1([0, T])$;
3. for a.e. $t \in [0, T]$, every $(x, y) \in \mathbb{R}^{2d}_{\not\triangle}$, for any $\varphi \in C_0(\mathbb{R}^d)$, it holds

$$\int_{\mathbb{R}^d} \varphi \mathrm{d}\rho_t = \int_{\mathbb{R}^d} \varphi \mathrm{d}\rho_0 + \frac{1}{2} \int_0^t \iint_{\mathbb{R}^{2d}_{\not\triangle}} \overline{\nabla} \varphi \mathrm{d}F^{\Phi}[\mu, \eta_s, \rho_s; V_s[\rho_s]] \mathrm{d}s \tag{14}$$

$$\eta_t(x, y) = \eta_0(x, y) + \int_0^t (\omega[\rho_s](s, x, y) - \eta_s(x, y)) \, \mathrm{d}s. \tag{15}$$

A.E., L. Mikolás, *On evolution PDEs on co-evolving graphs*, preprint arXiv:2310.10350.

Different time-scales

Graph slower: $\tau = \varepsilon t$

$$\begin{cases} \partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \\ \partial_t \eta_t = \varepsilon(\omega[\rho_t] - \eta_t) \\ \rho_0 \in \mathcal{M}_{TV}^{M}(\mathbb{R}^d), \ \eta_0 \in C_b(\mathbb{R}^{2d}_{\diagup}) \end{cases}, \qquad (\mathrm{Co-NCL_S})$$

Graph faster: $\tau = t/\varepsilon$

$$\begin{cases} \partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \\ \varepsilon \partial_t \eta_t(x,y) = -\eta_t(x,y) + \omega[\rho](t,x,y), \end{cases} \qquad (\mathrm{Co-NCL_F})$$

$\Rightarrow$

$$\partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \omega[\rho_t]; \rho_t, V_t[\rho_t]].$$

Take-home messages

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Thank you for your attention!