On a Boltzmann mean-field game model for knowledge growth

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Numerical methods for optimal transport, mean field games and multi-agent dynamics
Valparaiso, January 2024

Table of contents

Knowledge & economic growth

A Boltzmann mean field game model for knowledge growth

Special cases and generalisations of the model

Analysis of the original BMFG model

Endogenous growth theory

What initiates growth ?

Pareto tails

Knowledge diffusion

Limits to learning

From BMFG to MFGs

Knowledge & economic growth

Knowledge growth and the allocation of time[1]

Lucas and Moll's setting:

- *Consider a continuum of agents characterised by their knowledge level $z \in \mathcal{I}$ and the fraction of time $s = s(z,t) \in [0,1]$ devoted to learning.*
  Possible choices of $\mathcal{I}$: $\mathcal{I} = \mathbb{R}^+$ or $\mathcal{I} = [0, \bar{z}]$.
- *Agents either produce goods with the knowledge already obtained or meet others to increase their knowledge level.*

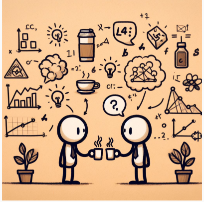


Figure 1: Image by Microsoft AI Image Creator

Knowledge exchange: When two agents with knowledge level z and z' meet, their post-meeting knowledge corresponds to

$$z^* = \max(z, z').$$

[1] R. E. Lucas Jr and B. Moll. Knowledge growth and the allocation of time. *Journal of Political Economics*, 2014

Agent dynamics

Boltzmann-type equat ion for the distribution of agents $f = f(z,t)$:

$$\partial_t f(z,t) = \underbrace{-\alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)\,\mathrm{d}y}_{\text{loss due to increase of knowledge level}} \quad \underbrace{+f(z,t)\int_0^z \alpha(s(y,t))f(y,t)\,\mathrm{d}y}_{\text{gain due to knowledge gain of agents with lower level}} \quad .$$

where $\alpha = \alpha(s)$ is the interaction probability.
For example:

$$\alpha(s) = \alpha_0 s^n, \quad n \in (0,1).$$

Individual productivity $y(t)$:

$$y(t) = \text{(time spent working)} \times \text{(knowledge level)} = (1 - s(z,t))z$$

Overall productivity: total earnings in an economy

$$Y(t) = \int_0^\infty [1 - s(z,t)]\, zf(z,t)\,\mathrm{d}z.$$

How much time should one spend on learning ?

Agent with knowledge level x maximises their earnings by choosing the optimal $s = s(z,t)$*:*

$$V(x,t') := \max_{s\in\mathcal{S}} \Big[\int_{t'}^{T}\int_0^\infty e^{-r(t-t')}(1-s(z,t))z\rho_x(z,t)\,\mathrm{d}z\,\mathrm{d}t\Big],$$

with $\mathcal{S} = \{s : \mathcal{I}\times[0,T]\to[0,1]\}$, $r\in\mathbb{R}^+$ *subject to*

$$\partial_t\rho_x(z,t) = -\alpha(s)\rho_x(z,t)\int_z^\infty f(y,t)\,\mathrm{d}y + f(z,t)\int_0^z \alpha(s)\rho_x(y,t)\,\mathrm{d}y$$

with $\rho_x(z,t') = \delta_x$.

Hamilton-Jacobi Bellman (HJB) equation for the value function $V = V(z,t)$:

$$\partial_t V(z,t) - rV(z,t) + \max_{s\in\mathcal{S}}\Big[(1-s(z,t))z + \alpha(s)\int_z^\infty [V(y,t)-V(z,t)]f(y,t)\,\mathrm{d}y\Big] = 0,$$

The full Boltzmann mean field game (BMFG) model

$$\partial_t f(z,t) = -\alpha(S(z,t)) f(z,t) \int_z^\infty f(y,t) dy + f(z,t) \int_0^z \alpha(S(y,t)) f(y,t)\, \mathrm{d}y.$$

$$\partial_t V(z,t) - rV(z,t) =$$

$$- \max_{s \in \mathcal{S}} \left[(1 - s(z,t))z - \alpha(s(z,t)) \int_z^\infty [V(y,t) - V(z,t)] f(y,t)\, \mathrm{d}y \right]$$

$$S(z,t) = \arg\max_{s \in \mathcal{S}} \left[(1 - s(z,t))z + \alpha(s(z,t)) \int_z^\infty [V(y,t) - V(z,t)] f(y,t)\, \mathrm{d}y \right],$$

$$f(z,0) = f_0(z),$$

$$V(z,T) = 0.$$

Special case $\alpha = \alpha_0$

BMFG system decouples:

$$\max_s \left[(1 - s(z,t))z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)] f(y,t)\, \mathrm{d}y \right]$$

is $S(z,t) = 0$.

Boltzmann equation can be rewritten using the cdf $F(z,t) = \int_0^z f(y,t) dy$*:*

$$\partial_t F(z,t) = -\alpha_0 \left(1 - F(z,t)\right) F(z,t).$$

Then the function $G(z,t) = 1 - F(z,t)$ *satisfies the Fisher-KPP-equation:*

$$\partial_t G(z,t) = \alpha_0 (1 - G(z,t)) G(z,t).$$

Recall: Diffusive Fisher-KPP equations has travelling wave solutions.

Original BMFG model is rather simplistic....

Limits to learning: *the larger the difference between the knowledge levels, the lower the learning rate.*

More general agent dynamics:

$$\begin{aligned}\partial_t f(z,t) = & f(z,t)\int_0^z \alpha(s(y,t))f(y,t)k\left(\frac{z}{y}\right)\,\mathrm{d}y \\ & -\alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)k\left(\frac{y}{z}\right)\,\mathrm{d}y.\end{aligned}$$

with an interaction function/learning rate k. For example

$$k\,(z,y) = \delta + (1-\delta)\left(\frac{z}{y}\right)^{-\kappa} \quad \text{where } \delta \in (0,1) \text{ and } \kappa > 0.$$

or $k(z,y) = \mu e^{-\kappa|z-y|}$ *with* $\mu, \kappa > 0$.

Different utility function: *Let* $p = (1 - s(z,t))z$, *then we can replace the linear utility* $U(p) = p$ *by*

the logarithmic one $U(p) = \ln p$ *or the isoelastic one* $U(p) = \dfrac{p^{1-\zeta}}{1-\zeta}$ *with* $\zeta \in (0,1)$.

Generalised BMFG model

Fully coupled system:

$$\begin{aligned}
\partial_t f(z,t) &= f(z,t)\int_0^z \alpha\left(s(y,t)\right) k(z,y) f(y,t)\,\mathrm{d}y \\
&\quad - \alpha\left(s(z,t)\right) f(z,t)\int_z^\infty k(y,z) f(y,t)\,\mathrm{d}y, \\
\partial_t V(z,t) - rV(z,t) &= -\max_{s\in\mathcal{S}}\left[U(p) + \alpha(s)\int_z^\infty \left(V(y,t)-V(z,t)\right) f(y,t) k(y,z)\,\mathrm{d}y\right], \\
S(z,t) &= \arg\max_{s\in\mathcal{S}}\left[U(p) + \alpha(s)\int_z^\infty \left(V(y,t)-V(z,t)\right) f(y,t) k(y,z)\,\mathrm{d}y\right], \\
f(z,0) &= f_0(z) \\
V(z,T) &= 0,
\end{aligned}$$

where f_0 is the initial distribution of agents.

If f_0 has compact support....

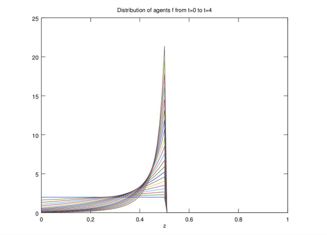
Consider the Boltzmann type equation for a given learning function $\alpha = \alpha(z,t)$:

$$\partial_t f(z,t) = -\alpha(z,t) f(z,t) \int_z^{\bar{z}} f(y,t)\, dy + f(z,t) \int_0^z \alpha(y,t) f(y,t) dy,$$
$$f(z,0) = f_0(z),$$

on the interval $\mathcal{I} = [0,\bar{z}]$, where $f_0 \in L^\infty(\mathcal{I})$ is a given probability density.

If $\alpha(z,t) \geq \underline{\alpha} > 0$ and $\tilde{z} = argmax_z supp(f_0)$, then

$$f(\cdot,t) \rightharpoonup^* \delta_{\tilde{z}}.$$



The full BMFG system

Assumptions:

(A1) Let the final data $V(\cdot, T)$ be non-negative and non-decreasing.

(A2) Let the interaction function satisfy:

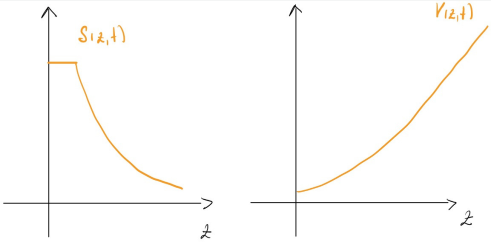
$$\alpha : [0,1] \to \mathbb{R}^+,\ \alpha \in C^\infty([0,1]),\ \alpha(0) = 0,\ \alpha'(0) = \infty,\ \alpha'' < 0 \text{ and } \alpha \text{ monotone.}$$

Theorem

Let $f_0(z) \in L^\infty(\mathcal{I})$ be a probability density and (A1) and (A2) be satisfied. If $\lim_{s\to 0} \frac{(\alpha')^3}{\alpha''} < \infty$, then the fully coupled Boltzmann mean field game system on $\mathcal{I} = \mathbb{R}^+$ has a unique local in time solution.

Idea of proof:

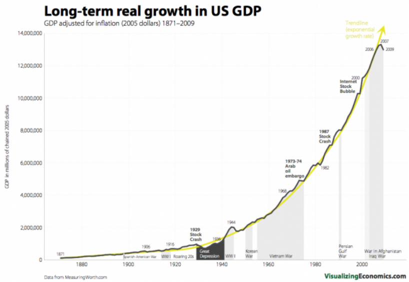
- *Fixed point argument.*
- *Relies heavily on qualitative properties of solutions, in particular*
  - *The value function V is a non-negative and non-decreasing function of the knowledge level z for all times $t > 0$.*
  - *The optimal learning time fraction S is a non-increasing function of the knowledge level z for all times $t > 0$.*



Endogenous growth theory

Endogenous growth theory

- *Endogenous growth theory proposes that economic growth is correlated to investments in human capital, innovation and knowledge.*
- *The 'performance' of economies is generally measured using the gross domestic product (GDP).*
- *The GDP of most developed countries has grown exponentially since World War II.*



- *Economists are interested in solutions which correspond to sustained/exponential growth - so-called balanced growth path (BGP) solutions.*

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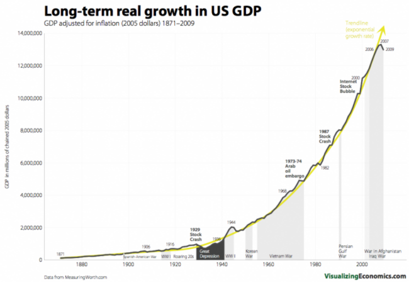


- *Economists are interested in solutions which correspond to sustained/exponential growth - so-called balanced growth path (BGP) solutions.*

Can we find BGP solutions to BMFG systems ?

Balanced growth path solutions (BGPs)

Assume there exists a growth parameter $\gamma \in \mathbb{R}^+$ and consider the re-scaling:

$$f(z,t) = e^{-\gamma t}\phi(ze^{-\gamma t}), \quad V(z,t) = e^{\gamma t} v(ze^{-\gamma t}) \text{ and } s(z,t) = \sigma(ze^{-\gamma t}).$$

Rescaled BMFG system in $(v, \phi, \sigma) = (v(x), \phi(x), \sigma(x))$ with $x = e^{-\gamma t} z$ reads as:

$$-\gamma\phi(x) - \gamma\phi'(x)x = \phi(x)\int_0^x \alpha(\sigma(y))\phi(y)\,dy - \alpha(\sigma(x))\phi(x)\int_x^\infty \phi(y)\,dy$$

$$(r-\gamma)v(x) + \gamma v'(x)x = \max_{\sigma\in\Xi}\left\{(1-\sigma)x + \alpha(\sigma)\int_x^\infty [v(y)-v(x)]\phi(y)\,dy\right\}$$

where $\Xi = \{\sigma : \mathbb{R}^+ \to [0,1]\}$ denotes the set of admissible controls.

Re-scaling results in exponential growth of the overall production Y(t):

$$Y(t) = e^{\gamma t}\int_0^\infty [1-\sigma(x)]x\phi(x)dx.$$

Existence of BGP solutions

Does such a growth parameter γ exist ?

Existence of BGP solutions

The initial commutative distribution function $F(z,0) = \int_0^z f_0(z)\,\mathrm{d}z$ *has a Pareto tail, if there exist constants* $k, \theta \in \mathbb{R}^+$ *such that*

$$\lim_{z\to\infty} \frac{1 - F(z,0)}{z^{-1/\theta}} = k. \tag{P}$$

Lemma

Let (P) be satisfied. Then solutions $F = F(z,t)$ *to the Boltzmann equation have a Pareto tail with the same decay rate* θ *for all times* $t \in [0, T]$.

Theorem

Let (P) be satisfied and $\alpha = \alpha_0$*. then there exists a unique BGP solution* $(\Phi, v, 0)$ *and a scaling constant* γ *given by*

$$\gamma = \alpha_0\theta \int_{\mathcal{I}} f_0(z)\,\mathrm{d}z, \quad \Phi(x) = \frac{1}{1 + kx^{-1/\theta}} \text{ with } \Phi(x) = \int_0^x \phi(y)\,\mathrm{d}y.$$

Existence of BGP solutions

Theorem

Let $r > \theta\alpha(1)$ and $\tilde{k} > 0$, then the BGP system has a non-trivial solution satisfying the Pareto-tail condition with $k = \frac{\gamma}{\theta}\tilde{k}$.

Idea of the proof: *Fixed points argument.*

- *Solve equations for (Φ, γ) given (v, S).*
- *Solve equations for (v, S) and given (Φ, γ).*

Challenge: *degenerate solution:*

$$\gamma = 0,\ v = \frac{x}{r} \text{ and } S \equiv 0 \Rightarrow \Phi(x) = 1 \text{ for } x > 0$$
$$\Rightarrow \phi(x) = \delta_0$$

Have to construct solutions Φ that satisfy a Pareto tail condition with some $k > 0$.

Knowledge diffusion initiates growth

Achdou et al. postulated that knowledge diffusion leads to BGPs (even in case of compactly supported f_0), with a growth parameter

$$\gamma = 2\sqrt{\nu \int_0^\infty \alpha(\sigma(y))\phi(y)dy}.$$

where σ corresponds to the diffusivity.
Later proven by Papanicolaou, et al. and Porretta and Rossi.

Special case $\alpha = \alpha_0$:

- *The Fisher KPP equation (with diffusion) admits travelling wave solutions*

$$G(z,t) = \Phi(z - \gamma t)$$

  with a minimal wave speed $\gamma = 2\sqrt{\nu\alpha_0}$.
- *Travelling waves correspond to BGP solutions (in logarithmic variables).*

[1] Y. Achdou, F.J. Buera, J.-M. Lasry, P.-L. Lions and B. Mol, Phil. Trans. Roy. Soc. A, 2014; G. Papnicolaou, L. Ryzhik and K. Velcheva, Nonlinearity (2021); A. Porretta and L. Rossi, Annales de l'Institute Henri Poincare C (2022)

Numerical simulations

- *Use an iterative solver for the time-dependent problem as well as the BGP system..*
- *We solve the systems on the interval* $\mathcal{I} = [0, \bar{z}]$ *with no-flux boundary conditions.*
- *To exclude degenerate BGP solutions we set*

$$\phi_0 = 0.$$

- *We use a finite difference discretization in space and approximate the integrals using the trapezoidal rule.*

The time-dependent solver

1. Given f_0 and S^k solve

$$\frac{1}{\tau}(f_i^{k+1} - f_i^k) - \frac{\nu}{h^2}(z^2_{i+\frac{1}{2}} f^{k+1}_{i+1} - (z^2_{i+\frac{1}{2}} + z^2_{i-\frac{1}{2}}) f_i^{k+1} + z^2_{i-\frac{1}{2}} f^{k+1}_{i+1}) + \frac{\nu}{h}(z_{i+\frac{1}{2}} f_i^{k+1} - z_{i-\frac{1}{2}} f^{k+1}_{i-1}) = g_1(f^k, S^k),$$

   for every $t^k = k\tau,\ k > 1$, where g_1 is the approximation of the gain/loss term.

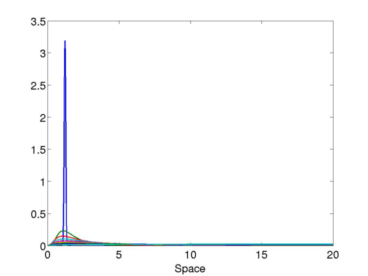
2. Update the maximizer S^k.
3. Given the evolution of the density f^k and the maximizer S^k solve the HJB equation

$$\frac{1}{\tau}(V_i^{k+1} - V^k) + \frac{\nu}{h^2} z_i^2 (V^k_{i+1} - 2V_i^k + V^k_{i-1}) + \frac{\nu}{h} z_i (V_i^k - V^k_{i-1}) - rV_i^k = g_2(S^{k+1}, f^{k+1}, V^{k+1}),$$

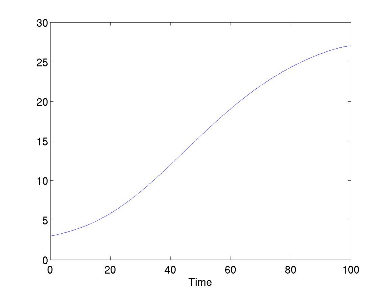
   backward in time, where g_2 is the approximation of the rhs.

4. Go to step (1) until convergence.

Time-dependent solutions



(a) Evolution of the agent density f.



(b) Overall productivity $Y(t) = \int (1 - s(z, t)) z f \, \mathrm{d}z$.

The BGP solver

1. Given ϕ^{n+1}, γ^n and σ^n solve

$$(r-\gamma^n)v_i^{n+1}+\frac{(\gamma^n-\nu)}{h}x_i(v_i^{n+1}-v_{i-1}^{n+1})-\frac{\nu x_i^2}{h^2}(v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}) = -q_2(\phi_{n+1},v^n,\sigma^n)$$

   where q_2 is an approximation of the gain/loss term.

2. Compute the maximum σ^{n+1} and update the growth parameter γ^{n+1} via

$$\gamma^{n+1}=2\big(\nu\int_{\mathcal{I}}\alpha(\sigma^{n+1}(y))\phi^{n+1}(y)\,\mathrm{d}y\big)^{\frac{1}{2}}.$$

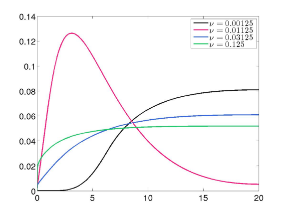
3. Given v^n,σ^n and γ^n solve

$$-(\gamma^n-\nu)\phi_i^{n+1}-\frac{(\gamma^n-\nu)}{h}x_i(\phi_{i+1}^{n+1}-\phi_i^{n+1})-(\Upsilon_i-\alpha(\sigma_i^n)(1-\Phi_i))\phi^{n+1} -\frac{\nu}{h^2}(x_{i+\frac12}^2\phi_{i+1}^{n+1}-(x_{i+\frac12}^2+x_{i-\frac12}^2)\phi_i^{n+1}+x_{i-\frac12}^2\phi_{i-1}^{n+1})=0,$$

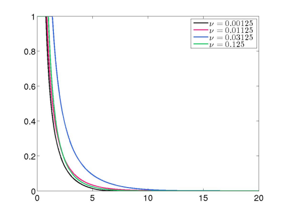
   as well as $(\phi_1^{n+1}+\phi_2^{n+1}+\dots\frac12\phi_N^{n+1})h=1$.
   The coefficients Υ_i and Φ_i correspond to approximations of the integrals $\int_0^{x_i}\alpha(\sigma(y))\phi(y)\,\mathrm{d}y$ and $\int_0^{x_i}\phi(y)\,\mathrm{d}y$.

4. Go to (1) until convergence.

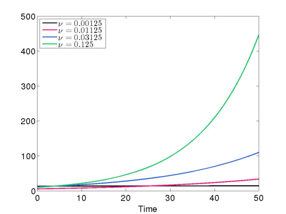
Simulation: BGP solutions in case of knowledge diffusion



(c) Agent distribution f for different diffusivities

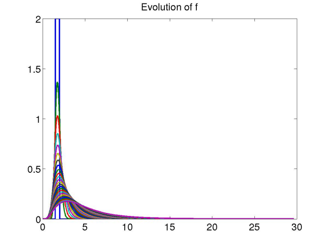


(d) Optimal learning time σ for different diffusivities.

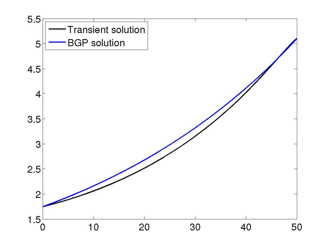


(e) Overall production Y.

Simulation: comparison original vs BGP with diffusivity $\nu = 0.05$



(f) Agent distribution f



(g) $Y(t) = \int (1-s) z f(z,t)\, \mathrm{d}z$

Recap

What we know so far

- *If F_0 has a Pareto tail, then BGPs to the original MFG exist.*
- *In the case of knowledge diffusion, then BGPs to the original MFG exist, and the growth depends on the diffusion parameter.*
- *Solutions to the original BMFG system have particular qualitative properties. For example*
  - *The value function V is a non-negative and non-decreasing function of the knowledge level z for all times $t > 0$.*
  - *The optimal learning time fraction S is a non-increasing function of the knowledge level z for all times $t > 0$.*

Recap

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  - *The optimal learning time fraction S is a non-increasing function of the knowledge level z for all times $t > 0$.*

What happens if we consider the more general BMFG model?

Limits to learning

General BMFG model

$$
\begin{aligned}
\partial_t f(z,t) &= f(z,t)\int_0^z \alpha\left(s(y,t)\right) k(z,y) f(y,t)\,\mathrm{d}y \\
&\qquad - \alpha\left(s(z,t)\right) f(z,t)\int_z^\infty k(y,z) f(y,t)\,\mathrm{d}y, \\
\partial_t V(z,t) - rV(z,t) &= -\max_{s\in\mathcal{S}}\left[U(p) + \alpha(s)\int_z^\infty \left(V(y,t)-V(z,t)\right) f(y,t) k(y,z)\,\mathrm{d}y\right], \\
S(z,t) &= \arg\max_{s\in\mathcal{S}}\left[U(p) + \alpha(s)\int_z^\infty \left(V(y,t)-V(z,t)\right) f(y,t) k(y,z)\,\mathrm{d}y\right], \\
f(z,0) &= f_0(z) \\
V(z,T) &= 0,
\end{aligned}
$$

with different learning rates k

$$
k\left(\frac{z}{y}\right) = \delta + (1-\delta)\left(\frac{z}{y}\right)^{-\kappa} \text{ where } \delta > 0,\ \kappa > 0,
$$

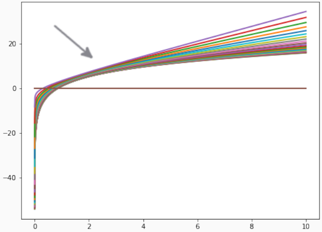
and utilities U:

$$
U(p) = p \text{ or } U(p) = \ln p \text{ or } U(p) = \frac{p^{1-\zeta}}{1-\zeta} \text{ with } \zeta \in (0,1).
$$

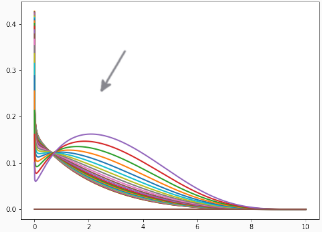
What we know so far for the general BMFG

- *Existence and uniqueness of solutions for small times for all bounded learning kernels k and for the linear as well as isoelastic utility U (not the logarithmic one).*
- *The value function V is non-decreasing for all choices of k and U considered.*
- *The optimal learning time S is non-increasing for the linear and isoelastic utility for certain parameter ranges of κ.*
- *Numerical simulation show non-monotonic behaviour in case of the logarithmic utility $U(p) = \ln p$.*

Simulation: non-monotonic behaviour for $U(p) = \ln p$



(h) Value function V



(i) Optimal time fraction S

Local MFGs

Consider a general BMFG with a localised kernel k, in particular

$$k(z, y) = \varepsilon^{-1} k_* \left(\frac{z - y}{\varepsilon} \right),$$

with k_* being symmetric and satisfying some more assumptions, and $\varepsilon \ll 1$.

Formal limit $\varepsilon \to 0$ gives a local mean field game (omitting higher order terms)

$$\partial_t f(z, t) = -\partial_z \left(f^2(z, t) \alpha(s(z, t)) \right),$$
$$\partial_t V(z, t) - rV(z, t) = -\max_{s \in \mathcal{S}} [U((1 - s)z) + \alpha(s(z, t)) f(z, t) \partial_z V(z, t)].$$

Local MFGs

Introduce the control variable

$$v(z,t) = \alpha(s(z,t))f(z,t), \quad s(z,t) = \alpha^{-1}\left(\frac{v(z,t)}{f(z,t)}\right)$$

with $\mathcal{V} := \{v : \mathbb{R}_+ \to \mathbb{R}_+\}$. Then

$$\partial_t f(z,t) + \partial_z(f(z,t)v(z,t)) = 0,$$
$$\partial_t V(z,t) - rV(z,t) = -\max_{v\in\mathcal{V}}\left[U\left(\left(1-\alpha^{-1}\left(\frac{v(z,t)}{f(z,t)}\right)\right)z\right) + v(z,t)\partial_z V(z,t)\right],$$

which can be written as an optimal control problem or potential mean field game

$$\max_{v\in\mathcal{V}} \int_0^T \int_0^\infty e^{-rt} w\left(\frac{v(z,t)}{f(z,t)}, z\right) f(z,t)\, dz \, \mathrm{d}t$$

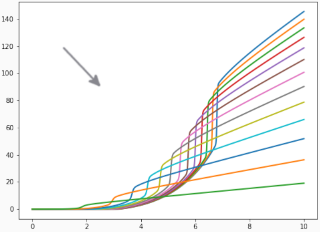
subject to

$$\partial_t f(z,t) + \partial_z\left(v(z,t)f(z,t)\right) = 0,$$

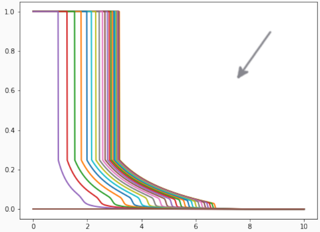
The function $w(\cdot,\cdot)$ depends on the utility U and has to satisfy:

$$w(p,z) - p\partial_p w(p,z) = -U((1-\alpha^{-1}(p))z), \quad \forall\, z \in \mathbb{R}_+.$$

Simulation: local MFG with linear utility



(j) V



(k) S

The end

And a To-Do List:

- *Existence of BGP solution to the general BMFG model.*
- *Numerical analysis for all systems.*
- *Existence of solutions to the local MFG.*
- *BGP solutions to local MFG?????*

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Thank you very much for you attention

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