Charaterizing the calmness property in convex semi-infinite optimization. Modulus estimates.

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1. Introduction and preliminaries

We consider the parameterized *linear optimization problem*:

$$P(c, a, b): \quad \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & a_t'x \leq b_t, \;\; t \in T \end{array}$$

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We assume that

- $x \in \mathbb{R}^n$ is the vector of decision variables
- y' denotes the transpose of $y \in \mathbb{R}^n$
- $c \in \mathbb{R}^n$, $(a,b) \equiv (a_t, b_t)_{t\in T} \in \left(\mathbb{R}^{n+1}\right)^T$

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Symbol ($\bigstar$) means "this result also holds for *semi-infinite problems*, i.e. when T is infinite".
If T is compact Hausdorff, $t \mapsto a_t \in \mathbb{R}^n$ and $t \mapsto b_t \in \mathbb{R}$ are continuous on T, the problem P is called *continuous*.

CANONICAL *vs* FULL PERTURBATIONS

- **Feasible set mappings,** $\mathcal{F} : \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{F}(a,b) := \left\{x \in \mathbb{R}^n : a_t' x \leq b_t, \text{ for all } t \in T\right\},$$

and $\mathcal{F}_{\overline{a}} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a}, b).$$

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- **Optimal set mappings,** $\mathcal{S} : \mathbb{R}^n \times \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$,

$$\mathcal{S}(c,a,b) := \{x \in \mathbb{R}^n \mid x \text{ is an optimal solution for } P(c,a,b)\},$$

and $\mathcal{S}_{\overline{a}} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{S}_{\overline{a}}(c,b) := \mathcal{S}(c,\overline{a},b) \text{ (\textit{canonical perturbations})}.$$

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$$\|(c,a,b)\| := \max\left\{\|c\|_*, \sup_{t\in T}\|(a_t,b_t)\|\right\},$$

where:

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(Recall that $\|u\|_* := \max_{\|x\|\leq 1} |u'x|$).

- For canonical perturbations, when $a = \bar{a}$, $\mathbb{R}^n \times \mathbb{R}^T$ is also endowed with the corresponding supremum norm .

$q-$order error bounds of funtions

Definition

Given the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a metric space X, a point $\bar{x} \in [f \leq 0]$ and a number $q > 0$, we say that f admits a $q-$*order local error bound* at $\bar{x}$, if $\exists$ $\kappa \geq 0$ and $\exists$ a neighb. U of $\overline{x}$ such that

$$d(x, [f \leq 0]) \leq \kappa [f(x)]_+^q, \quad \forall x \in U. \tag{1}$$

If $q = 1$, we say that f admits a *local error bound* at $\bar{x}$.

The infimum of all κ in (1) is called the *modulus of q-order error bounds of f at $\bar{x}$*, and it is denoted by $\mathrm{Er}_q f(\bar{x})$.
The absence of q-order error bounds corresponds to $\mathrm{Er}_q f(\bar{x}) = +\infty$.

Hölder calmness of mappings

X, Y metric spaces (distances in X and Y are denoted by d),

Definition

$\mathcal{M} : Y \rightrightarrows X$ is *q-order calm* , $q \in]0, 1]$, at $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$ if $\exists U$ neighb. of $\bar{x}$, $\exists V$ neighb. of $\bar{y}$, $\exists \kappa \geq 0$ such that

$$d\left(x, \mathcal{M}\left(\bar{y}\right)\right) \leq \kappa d\left(y, \bar{y}\right)^{q}, \ \forall y \in V, \ \forall x \in \mathcal{M}\left(y\right) \cap U. \qquad (2)$$

$\mathcal{M}$ is *calm* if $q = 1$.
The infimum of all $\kappa \geq 0$ for which (2) holds is called the *q-order calmness modulus* of $\mathcal{M}$ at $(\bar{y}, \bar{x})$; it is

$$\operatorname{clm}_{\mathrm{q}} \mathcal{M}(\bar{y}, \bar{x}) = \limsup_{\substack{y \to \bar{y}, \\ x \to \bar{x}, x \in \mathcal{M}(y)}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})^{q}}.$$

If $\operatorname{clm}_{\mathrm{q}} \mathcal{M}(\bar{y}, \bar{x}) = +\infty$, $\mathcal{M}$ is *not* q*−calm* at $(\bar{y}, \bar{x})$.

Calmness under canonical pertubations

Theorem (Robinson, 1981)

If $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is polyhedral ($\operatorname{gph} \mathcal{M}$ is the finite union of polyhedral sets), then $\mathcal{M}$ is calm at any $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$.

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Corollary

If T is finite, then $\mathcal{F}_{\bar{a}}$ and $\mathcal{S}_{\bar{a}}$ are calm at any element of their graphs.

Remark $\mathcal{S}_{\bar{a}}$ is calm at any point of $\operatorname{gph} \mathcal{S}_{\bar{a}}$ as Karush-Kuhn-Tucker conditions allow us to express the graph of $\mathcal{S}_{\bar{a}}$ as the finite union of polyhedral sets. This is no longer the case for $\mathcal{S}$ in the framework of perturbations of all data.

What about the continuous problem P? For the continuous system $(\overline{a}, \overline{b})$, consider the *supremum function*

$$\overline{s}(x) := \max\{\overline{a}_t' x - \overline{b}_t,\ t \in T\}. \tag{3}$$

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For any $x \in \mathbb{R}^n$,

$$\partial\overline{s}\,(x) = \operatorname{conv}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\},$$

where

$$T_{(\overline{a},\overline{b})}(x) := \{t \in T :\ \overline{a}_t' x - \overline{b}_t = \overline{s}\,(x)\}.$$

Due to the continuity of $\overline{s}\,(\cdot)$, $\overline{x} \in \operatorname{bd} \mathcal{F}_{\overline{a}}(\overline{b}) \ \Rightarrow \overline{s}\,(\overline{x}) = 0$.

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Due to the continuity of $\overline{s}(\cdot)$, $\overline{x} \in \operatorname{bd} \mathcal{F}_{\overline{a}}(\overline{b}) \ \Rightarrow \overline{s}(\overline{x}) = 0.$

Proposition (CLPT'14)

$\mathcal{F}_{\overline{a}}$ is calm at $(\overline{b}, \overline{x})$ if and only if $\overline{s}$ has a local error bound at $\overline{x}$; i.e., there exist $\kappa \geq 0$ and a neighborhood U of $\overline{x}$ such that

$$d(x, [\overline{s} \leq 0]) \leq \kappa [\overline{s}(x)]_+ \text{, for all } x \in U.$$

Theorem

If P *is a continuous problem and* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$, *TFAE:*

(i) $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b}, \overline{x})$

(ii) $\alpha := \liminf_{x \to \overline{x},\ \overline{s}(x) > 0} d_* \left(0_n, \partial \overline{s}\left(x\right)\right) > 0$

(iii) $\beta := \liminf_{x \to \overline{x},\ \overline{s}(x) > 0} \sup_{u \neq x} \frac{\left[\overline{s}\left(x\right) - \left[\overline{s}\left(u\right)\right]_+\right]_+}{d\left(x, u\right)} > 0$

Moreover, we have

$$\mathrm{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \alpha^{-1} = \beta^{-1}.$$

Remarks $(i) \Leftrightarrow (ii)$ *can be traced out from Azé and Corvellec'*04*;* $(i) \Leftrightarrow (iii)$ *from Fabian, Henrion, Kruger and Outrata'*12.

- P satisfies the *Abadie CQ* (ACQ) *around* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$ if $\exists$ neighb. U of $\overline{x}$ such that

  $$\mathcal{N}(\mathcal{F}_{\overline{a}}(\overline{b}), x) = \overline{\mathrm{cone}}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\} \text{ at any } x \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b}) \cap U,$$

  where $\mathrm{cone}\, A$ is the convex cone generated by A.
- P verifies the *uniform dual boundedness condition* (UDB condition) *around* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$ if $\exists\, M > 0$ and a neighb. U of $\overline{x}$ such that

  $$\mathrm{cone}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\} \cap \mathbb{B}_* \subset [0, M]\partial\overline{s}(x),\ \forall x \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b}) \cap U.$$

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Theorem (CLPT'14 Th. 3)

Let $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$*. Then* $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b}, \overline{x})$ *if and only if* P *satisfies ACQ and UDB around* $\overline{x}$.

Remark This result is inspired in *Zheng and Ng*'03. ACQ and UDB are independent properties.

Calmness modulus of the feasible set mapping

Fix $(\overline{a},\overline{b}) \in \left(\mathbb{R}^{n+1}\right)^T$, and associated with $\overline{x} \in \mathcal{F}(\overline{a},\overline{b})$, consider the family of subsets in $T_{(\overline{a},\overline{b})}(\overline{x})$:

$$\mathcal{D}\left(\overline{x}\right) := \left\{ D \subset T_{(\overline{a},\overline{b})}(\overline{x}) \;\middle|\; \begin{array}{c} \text{There exists } d \text{ verifying :} \\ \left\{ \begin{array}{ll} \overline{a}_t' d = 1, & t \in D, \\ \overline{a}_t' d < 1, & t \in T_{(\overline{a},\overline{b})}\left(\overline{x}\right) \setminus D \end{array} \right\} \end{array} \right\}$$

Theorem (CLPT'14, Ths. 4 and 5)

(i) If T is finite, $\operatorname{clm}\mathcal{F}_{\overline{a}}(\overline{b},\overline{x}) = \max\limits_{D \in \mathcal{D}(\overline{x})} \left(d_*(0_n, \operatorname{conv}\{\overline{a}_t, t \in D\})\right)^{-1}$

(ii) $(\bigstar)$ $\operatorname{clm}\mathcal{F}((\overline{a},\overline{b}),\overline{x}) = (\|\overline{x}\| + 1)\operatorname{clm}\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$

Calmness under full perturbations

Theorem (CLPT'14, Cor. 2 ($\bigstar$))

Let $((\overline{a},\overline{b}),\overline{x}) \in \operatorname{gph} \mathcal{F}$*; TFAE:*
(i) $\mathcal{F}$ *is calm at* $((\overline{a},\overline{b}),\overline{x})$*;*
(ii) $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b},\overline{x})$*.*

Calmness under full perturbations

Theorem (CLPT'14, Cor. 2 (★))

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(i) $\mathcal{F}$ *is calm at* $((\overline{a},\overline{b}),\overline{x})$;
(ii) $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b},\overline{x})$.

Theorem (CHaPT'16, Th. 4.1)

Assume that T *is finite and* $\mathcal{S}(\overline{c},\overline{a},\overline{b}) = \{\overline{x}\}$. *The following are equivalent:*
(i) $\mathcal{S}$ *is calm at* $((\overline{c},\overline{a},\overline{b}),\overline{x})$;
(ii) *Either Slater holds at* $(\overline{a},\overline{b})$ *or* $\mathcal{F}(\overline{a},\overline{b}) = \{\overline{x}\}$;
(iii) $0_n \notin \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{(\overline{a},\overline{b})}(\overline{x})\right\}$.

(Slater at $(\overline{a},\overline{b})$: there exists $\widehat{x} \in \mathbb{R}^n$ such that $\overline{a}_t'\widehat{x} < \overline{b}_t$, $t \in T$)

Hölder calmness of the optimal set in convex SIP

Consider the following *convex SIP problem*:

$$P(c,b): \quad \begin{array}{ll} \text{minimize} & f(x) + c'x \\ \text{subject to} & g_t(x) \leq b_t,\ t \in T, \end{array}$$

where $c, x \in \mathbb{R}^n$, T is a *compact set*, $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are *convex functions* such that $(t,x) \mapsto g_t(x)$ is *continuous* on $T \times \mathbb{R}^n$, and $t \mapsto b_t$ is *continuous* on T.
Also now, the pair $(c,b) \in \mathbb{R}^n \times \mathcal{C}(T,\mathbb{R})$ is the parameter to be perturbed, and the parameter space $\mathbb{R}^n \times \mathcal{C}(T,\mathbb{R})$ is endowed with the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\},$$

where now $\mathbb{R}^n$ is equipped with the Euclidean norm $\|\cdot\|$ and $\|b\|_\infty := \max_{t\in T} |b_t|$.

We deal with the optimal set mapping

$$\mathcal{S} : (c, b) \mapsto \{x \in \mathbb{R}^n \mid x \text{ is optimal for } P(c, b)\},$$

with $(c, b) \in \mathbb{R}^n \times \mathcal{C}(T, \mathbb{R})$.
In the case that $\overline{c}$ is fixed, $\mathcal{S}$ reduces to the partial optimal solution mapping $\mathcal{S}_{\overline{c}} : \mathcal{C}(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{S}_{\overline{c}}(b) = \mathcal{S}(\overline{c}, b).$$

Now, the feasible set mapping is given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid g_t(x) \leq b_t, t \in T\},$$

and the set of active indices at $x \in \mathcal{F}(b)$ by

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\}.$$

Definition

The problem $P(c,b)$ satisfies the *Slater constraint qualification* if there exists $\hat{x}$ such that $g_t(\hat{x}) < b_t$ for all $t \in T$.

The following result plays a crucial role in our analysis.

Proposition

Let $(\bar{c}, \bar{b}) \in \mathbb{R}^n \times \mathcal{C}(T, \mathbb{R})$ *and assume that* $P(\bar{c}, \bar{b})$ *satisfies the Slater condition. Then* $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$ *if and only if the Karush-Kuhn-Tucker (KKT) conditions hold, i.e.,*

$$\bar{x} \in \mathcal{F}(\bar{b}) \quad \textit{and} \quad -(\partial f(\bar{x}) + \bar{c}) \bigcap \operatorname{cone}\left(\bigcup_{t \in T_{\bar{b}}(\bar{x})} \partial g_t(\bar{x}) \right) \neq \emptyset.$$

$\operatorname{cone}(X)$ is the conical convex hull of X; always contains 0_n, entailing $\operatorname{cone}(\emptyset) = \{0_n\}$.

We use the *level set mapping* $\mathcal{L} : \mathbb{R} \times \mathcal{C}(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$

$$\mathcal{L}(\alpha, b) := \{x \in \mathbb{R}^n \mid f(x) + \bar{c}'x \leq \alpha;\ g_t(x) \leq b_t, t \in T\}$$

and the *supremum function* $h : \mathbb{R}^n \to \mathbb{R}$

$$h(x) := \sup\{f(x) + \bar{c}'x - \left(f(\bar{x}) + \bar{c}'\bar{x}\right);\ g_t(x) - \bar{b}_t,\ t \in T\}.$$

With $t_0 \notin T$, we define

$$\overline{T} := T \cup \{t_0\},\ g_{t_0}(x) := f(x) + \bar{c}'x \text{ and } \bar{b}_{t_0} := f(\bar{x}) + \bar{c}'\bar{x},$$

and obviously,

$$h(x) = \sup\{g_t(x) - \bar{b}_t,\ t \in \overline{T}\}.$$

$\overline{T}$ is compact (as t_0 is an isolated point in $\overline{T}$), the functions $(t, x) \mapsto g_t(x)$ is continuous on $\overline{T} \times \mathbb{R}^n$, $b \in \mathcal{C}(\overline{T}, \mathbb{R})$.

Given the supremum function

$$h(x) := \sup \left\{ \underbrace{f(x) + \bar{c}'x}_{g_{t_0}(x)} - \underbrace{(f(\bar{x}) + \bar{c}'\bar{x})}_{\bar{b}_{t_0}};\ g_t(x) - \bar{b}_t,\ t \in T \right\},$$

and the active set at x

$$\overline{T}(x) := \{t \in \overline{T} :\ h(x) = g_t(x) - \bar{b}_t\},$$

we have $t_0 \in \overline{T}(\bar{x})$, and

$$\partial h(x) = \operatorname{conv}\left(\bigcup\nolimits_{t \in \overline{T}(x)} \partial g_t(x)\right). \tag{4}$$

Since

$$\mathcal{L}(\alpha, b) := \{x \in \mathbb{R}^n \mid f(x) + \bar{c}'x \leq \alpha;\ g_t(x) \leq b_t, t \in T\}$$

and $((\bar{c}, \bar{b}), \bar{x}) \in \operatorname{gph}(\mathcal{S})$,

$$\mathcal{S}(\bar{c}, \bar{b}) = [h = 0] = [h \leq 0] = \mathcal{L}(f(\bar{x}) + \langle \bar{c}, \bar{x} \rangle, \bar{b}).$$

The following lemma provides a uniform boundedness result.

Lemma

Let $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$ *and assume that* $P(\bar{c},\bar{b})$ *satisfies Slater. Then, there exist* $M > 0$ *and neighborhoods* U *of* $\bar{x}$ *and* V *of* $(\bar{c},\bar{b})$ *such that, for all* $(c,b) \in V$ *and all* $x \in \mathcal{S}(c,b) \cap U$*, we have*

$$-(\partial f(x) + c) \bigcap [0,M]\,\mathrm{conv}\left(\bigcup_{t\in T_b(x)} \partial g_t(x)\right) \neq \emptyset. \quad (5)$$

Our approach strongly relies on the following proposition:

Proposition

Let $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$. *Then the following statements are equivalent:*

(i) $\mathcal{L}$ *is* q *-order calm at* $((f(\bar{x})+\langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{L})$*;*
(ii) $\liminf_{x\to\bar{x},h(x)\downarrow 0} h(x)^{q-1} d(0,\partial h(x)) > 0$.

Moreover,

$$\begin{aligned} \mathrm{Er_q} h(\bar{x}) &= \mathrm{clm_q}\, \mathcal{L}((f(\bar{x})+\langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x}) = \\ &= \left\{ \liminf_{x\to\bar{x},h(x)\downarrow 0} h(x)^{q-1} d(0,\partial h(x)) \right\}^{-1}. \end{aligned}$$

The following theorem constitutes a Hölder convex counterpart of Theorem 3.1 in *Cánovas-Et-Al'14* for the linear case.

Theorem (Theorem 4.7 in *Kruger, L, Yang, Zhu'19*)

Let $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$ *and assume that* $P(\bar{c}, \bar{b})$ *satisfies the Slater condition. Consider the following statements:*

(i) $\mathcal{S}$ *is* q*-order calm at* $((\bar{c}, \bar{b}), \bar{x})$;
(ii) $\mathcal{S}_{\bar{c}}$ *is* q*-order calm at* $(\bar{b}, \bar{x})$;
(iii) $\mathcal{L}$ *is* q*-order calm at* $((f(\bar{x}) + \langle \bar{c}, \bar{x} \rangle, \bar{b}), \bar{x})$;
(iv) h *has a* q*-order local error bound at* $\bar{x}$.
Then $(iii) \Leftrightarrow (iv) \Rightarrow (i) \Rightarrow (ii)$ *hold.*

In addition, if f *and* g_t *are linear, then* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.

In the convex setting, $(ii) \Rightarrow (iii)$ could fail:

$$P(0,0): \quad \begin{array}{ll} \text{minimize} & x^2 \\ \text{subject to} & x \leq 0, \end{array}$$

Take $\bar{c} = 0$, $\bar{b} = 0$, and $\bar{x} = 0$. Then $\mathcal{S}_{\bar{c}}(\bar{b}) = \{0\}$ and $h(x) = \sup\{x^2, x\}$.

a) Given $q \in (1/2, 1]$, it is easy to verify that

$$\liminf_{x \to \bar{x}, h(x) \downarrow 0} h(x)^{q-1} d(0, \partial h(x)) = 0$$

and, by Proposition 3, $\mathcal{L}$ is not q-order calm at $((0,0),0) \in \mathrm{gph}(\mathcal{L})$.

b) On the other hand, we have

$$\mathcal{S}_{\bar{c}}(b) = \min\{0, b\}, \text{ for all } b \in (-1, 1). \tag{6}$$

Since $\|b\| \leq \|b\|^{\frac{2}{3}}$ $\forall b \in (-1, 1)$, it follows from (6)

$$d(x, \mathcal{S}_{\bar{c}}(\bar{b})) \leq \|b - \bar{b}\|^{\frac{2}{3}} \quad \forall x \in \mathcal{S}_{\bar{c}}(b) \cap (-1, 1) \text{ and } b \in (-1, 1), \tag{7}$$

i,e., $\mathcal{S}_{\bar{c}}$ is $2/3$-order calm at $(0, 0)$.

To provide a lower estimate for $\mathrm{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$, we associate with $(b,x) \in \mathrm{gph}\,\mathcal{S}_{\bar{c}}$ the family of *KKT index sets*

$$\mathcal{M}_b(x) := \left\{ D \subset T_b(x) \;\middle|\; \begin{array}{c} -(\partial f(x) + c) \bigcap \mathrm{cone}\left(\bigcup_{t \in D} \partial g_t(x)\right) \neq \emptyset \\ \text{and } D \text{ is minimal for the inclusion order} \end{array} \right\}$$

With any $D \in \mathcal{M}_{\bar{b}}(\bar{x})$, we associate the function $f_D : \mathbb{R}^n \to \mathbb{R}$

$$f_D(x) := \sup\{g_t(x) - \bar{b}_t, t \in T;\ -g_t(x) + \bar{b}_t, t \in D\}.$$

Theorem

Let $\mathcal{S}(\bar{c},\bar{b}) = \{\bar{x}\}$ and assume that $P(\bar{c},\bar{b})$ satisfies Slater. Then

$$\limsup_{\substack{x \to \bar{x},\, b \to \bar{b} \\ x \in \mathcal{F}(b)}} \frac{\|x - \bar{x}\|}{\|b - \bar{b}\|_\infty^q} \geq \left(\inf_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} \liminf_{\substack{x \to \bar{x} \\ f_D(x) > 0}} f_D(x)^{q-1} d(0, \partial f_D(x)) \right)^{-1}.$$

Finally, we consider the linear case, i.e. $f = 0$, and $g_t(x) = a_t' x$ for all $t \in T$, and $t \mapsto a_t \in \mathbb{R}^n$ is continuous on T.

Proposition

Let $\mathcal{S}(\bar{c}, \bar{b}) = \{\bar{x}\}$ and assume that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Then the following estimates hold

$$\begin{aligned} \operatorname{clm_q} \mathcal{S}((\bar{c}, \bar{b}), \bar{x}) &\geq \operatorname{clm_q} \mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \\ &\geq \left(\inf_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} \liminf_{\substack{x \to \bar{x} \\ f_D(x) > 0}} f_D(x)^{q-1} d(0, \partial f_D(x)) \right)^{-1} \\ &= \sup_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} \operatorname{Er_q} f_D(\bar{x}). \end{aligned}$$

If T is finite, the inequalities become equalities.

$q-$isolated calmness

Definition

Given a set valued mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces Y and X and a number $q > 0$, we say that $\mathcal{M}$ is *q-order isolated calm* at $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$ if there exist a number $\kappa > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, \bar{x}) \leq \kappa d(y, \bar{y})^q, \quad \forall y \in V \text{ and } x \in \mathcal{M}(y) \cap U. \tag{8}$$

(8) implies that $\mathcal{M}(\bar{y}) \cap U = \{\bar{x}\}$.

The infimum of all $\kappa > 0$ in (8) is called the *q-order isolated calmness modulus* of $\mathcal{M}$ at $(\bar{y}, \bar{x})$ and is also denoted by $\operatorname{clm}_q \mathcal{M}(\bar{y}, \bar{x})$.
The $1-$isolated calmness of $\mathcal{M}$ at $(\bar{y}, \bar{x})$ is equivalent to the *strong metric subregularity* of $\mathcal{M}^{-1}$ at $(\bar{x}, \bar{y})$.

Definition

Given a set valued mapping $\mathcal{M} : Y \rightrightarrows X$ between normed spaces Y and X, its *$q-$order graphical derivative* at $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$ is a set-valued mapping $D_q\mathcal{M}(\bar{y}, \bar{x}) : Y \rightrightarrows X$ defined by

$$D_q\mathcal{M}(\bar{y}, \bar{x})(y) = \liminf_{t \downarrow 0,\ y' \to y} \frac{\mathcal{M}(\bar{y} + ty') - \bar{x}}{t^q}.$$

The *outer norm* of the mapping $D_q\mathcal{M}(\bar{y}, \bar{x})$ is

$$\left\| D_q\mathcal{M}(\bar{y}, \bar{x}) \right\| := \sup_{(y,x) \in \operatorname{gph} D_q\mathcal{M}(\bar{y}, \bar{x})} \frac{\|x\|}{\|y\|^q}.$$

Theorem

Let $\mathcal{S}_{\bar{c}}(\bar{b}) = \{\overline{x}\}$ *and assume that* $P(\bar{c}, \bar{b})$ *satisfies the Slater qualification. Consider the following statements:*

(i) $\mathcal{S}_{\bar{c}}$ *is q-order isolated calm at* $(\bar{b}, \bar{x})$*;*

(ii) h *has a* $(1/q)$*-order sharp minimum at* $\bar{x}$*; i.e., there exists* $\alpha > 0$ *such that*

$$h(x) \geq h(\overline{x}) + \alpha \|x - \overline{x}\|^{1/q}, \text{ for all } x \text{ near to } \overline{x}.$$

(iii) h *has a q-order local error bound at* $\bar{x}$*.*

Theorem (cont'd)

(iv) If $p = (1/q) - 1$*, then* $D_p \partial h(\bar{x}, 0_n)$ *is* $q-$*order positively definite; i.e.* $\exists c > 0$ *such that*

$$c \|x\|^2 \leq \langle u, x \rangle^{2q}, \forall x \in \mathbb{R}^n, \ \forall u \in \liminf_{t \downarrow 0,\ x' \to x} \frac{\partial h(\bar{x} + tx')}{t^p}.$$

(v) $\left\| D_q \mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \right\| < +\infty.$

Then $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (i) \Rightarrow (v)$.

If $P(\bar{c}, \bar{b})$ *is linear and* T *is finite, all these statements are equivalent, and then*

$$\mathrm{clm_q}\, \mathcal{S}_{\bar{c}}(\bar{b}, , \bar{x}) = \left\| D_q \mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \right\|.$$

Inspired in *Kruger,L.,Yang, Zhu'20, Aragón-Artacho, Geoffroy'14, Zheng, Ng'15*, and *Mordukhovich, Nghia'15.*

Example

Consider the linear SIP, in $\mathbb{R}^2$,

$$P(\bar{c}, b): \quad \text{Inf } x_1 \quad \text{s.t. } (\cos t)\, x_1 + (\sin t)\, x_2 \leq b_t,\ t \in [0, 2\pi].$$

Take for the nominal problem $\overline{b}_t = 1$, *for all* t. *Then,* $\overline{x} = (-1, 0)$ *is the unique optimal solution of* $P\left(\overline{c}, \overline{b}\right)$.
In Cánovas, Dontchev, L, Parra'09 it is shown that $\mathcal{S}_{\overline{c}}$ *is not isolated calm at* $(\bar{b}, \bar{x})$, *but* $\mathcal{S}_{\overline{c}}$ *is* q*-isolated calm for* $q = 1/2$:
(a) Observe first that

$$h(x) = \max\{x_1 + 1, \|x\| - 1\}.$$

Example

(b) Then, if

$$\begin{aligned} A_1(\varepsilon) &:= \{x \in \mathbb{R}^n : \|x-\bar{x}\|_\infty \leq \varepsilon \text{ and } x_1+1 > \|x\|-1\}, \\ A_2(\varepsilon) &:= \{x \in \mathbb{R}^n : \|x-\bar{x}\|_\infty \leq \varepsilon \text{ and } x_1+1 < \|x\|-1\}, \\ A_3(\varepsilon) &:= \{x \in \mathbb{R}^n : \|x-\bar{x}\|_\infty \leq \varepsilon \text{ and } x_1+1 = \|x\|-1\}, \end{aligned}$$

we get

$$\liminf_{x\to\bar{x},\, x\neq\bar{x}} h(x)^{q-1} d(0, \partial h(x)) = \lim_{\varepsilon\downarrow 0} \min_{i=1,2,3} \inf_{x\in A_i(\varepsilon)} h(x)^{q-1} d(0_2, \partial h(x))$$

$$= \lim_{\varepsilon\downarrow 0} \min_{i=1,2,3} \left(\varepsilon^{-1/2}, \left(\sqrt{1+2\varepsilon+2\varepsilon^2} - 1 \right)^{-1/2}, 2(\varepsilon^2+4)^{-1/2} \right)$$

$$= 1.$$

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Antecedents

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