Enlargements of the Moreau-Rockafellar subdifferential

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Outline

1. Enlargement of multifunctions

2. The Sup* and Sup-differential

3. The Symmetric Subdifferential

 

M. Abassi, A. K. Kruger, M.T, ***Enlargements of the Moreau-Rockafellar Subdifferential,*** *https://hal.archives-ouvertes.fr/hal-02484321*

Paper is dedicated to Terry Rockafellar, the ruler of Convex Analysis and Endolandia on the occasion of his 85th birthday.

Outline

Starting point of this research: *the example of the Rainwater function (see also Borwein, Phelps, Lucchetti) for which the Moreau-Rockafellar subdifferential is empty at every point of its domain.*

J. Rainwater : Yet more on the differentiability of convex functions. Proc. Amer. Math. Soc., 103(3):773–778, 1988.

$$X := \ell^2(\mathbb{N}) \quad \text{and} \quad C := \{x \in \ell^2(\mathbb{N}) : |x_n| \leq 2^{-n}, \quad n = 1, 2, \cdots\}$$

$$f(x) := \sum -(2^n + x_n)^{\frac{1}{2}} \quad \text{if} \quad x \in C$$

$$f(x) = +\infty \quad \text{elsewhere}$$

The Rainwater function leads to consider the question of enlargement of the Moreau-Rockafellar subdifferential.

What do we mean by an enlargement of a set-valued mapping and in particular of a monotone operator and of a subdifferential?

Revalski, J. P., Théra, M. : *Enlargements and sums of monotone operators, Nonlinear Anal., 48, 2002, 4, Ser. A: Theory Methods, 505 – 519.*

Burachik, R. S.; Iusem, A. N.; Svaiter, B. F.: *Enlargement of monotone operators with application to variational inequalities, Set-Valued Anal. 5(2), 159 – 180, 1997.*

Martinez-Legaz, J.-E., Théra, M. : *ε-subdifferentials in terms of subdifferentials, Set-Valued Anal., 4,1996, 4, 327 – 332.*

We will first give some hints of this question and then focus on possible extensions of the Moreau-Rockafellar subdifferential.

Many mathematical models coming from various problems can be expressed as

(OP)	*Find* $x \in \operatorname{dom} T$ *such that* $0 \in T(x)$.

When T is the subdifferential of some extended-real-valued convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, then **(OP)** becomes the Fermat rule:

$$0 \in T(\bar{x}) \iff \bar{x} \text{ minimizes } f$$

which is a central fact in optimization theory.

In some situations it is convenient to consider a set-valued mapping $T' : X \rightrightarrows X^*$ such that $T(x) \subset T'(x)$ for all $x \in X$ (called an enlargement of T) and staying " close" to T, in a sense that will be clarified later on and to consider the auxiliary problem:

(EP)	*Find an enlargement* T' *of* T *such that* $0 \in T'(x)$ *for all* $x \in \operatorname{dom} T'$

Solutions of the latter problem can serve as approximate solutions to the original one.

Example 1: The ε-subdifferential

$$\partial f_\epsilon(x) = \{x^\star \in X^\star : \text{ such that } \langle x^\star, y - x\rangle \le f(y) - f(x) + \epsilon \quad \forall y \in X\}$$

It **quantifies** the approximation error intrinsic to numerical computation when one is interested in the solution of a convex nonsmooth optimization problem. It extends Fermat rule and approximates the convex subdifferential through the Brondsted-Rockafellar Theorem.

It is known that when f is lsc then $\partial_\varepsilon f(x)$ is nonempty, weak*−closed and convex for every x.

Calculus rules exists but are difficult to apply!

Note that the ε-subdifferential can be viewed as an approximation of the subdifferential:

Theorem (Martinez-Legaz & MT)

$$\partial_\varepsilon f(x) = \{x^\star : \langle x^\star, x - x_0\rangle + \sum_{i=0}^{m-1} \langle x_i^\star, x_i - x_{i+1} + \langle x_m^\star, x_m - x\rangle \ge -\varepsilon\}$$

for all $(x_i, x_i^\star)$ in the graph of ∂f and every $(i = 0, 1 \cdots, m)$

Example 2: Monotone operators

Inspired by the notion of ε-subdifferential of a proper lsc convex function $f : X \to \mathbb{R}_\infty$, Revalski & Théra defined an enlargement :

given a monotone operator $A : X \rightrightarrows X^*$ and $\varepsilon \geq 0$, they defined $A_\varepsilon : X \to X^\star$ by

$$A^\varepsilon x := \{x^\star \in X^\star : \langle y - x, x^\star - y^\star \rangle \geq -\varepsilon \quad \text{for any} \quad (y, y^\star) \in Gr(A)\}.$$

A^ε is always with convex and w*-closed images for any $\varepsilon \geq 0$ and that indeed, due to the monotonicity of A, it is an enlargement of A:

$$Ax \subset A^\varepsilon x \text{ for any } \varepsilon \geq 0 \text{ and } x \in X.$$

$(\partial f)^\varepsilon$ is larger than the ε-subdifferential and the inclusion could be strict (Martinez Legaz-MT):

$$f(x) = x^2 \quad 0 \notin \partial_{\frac{1}{2}} f(1) \quad \text{but} \quad 0 \in (\partial f)^{\frac{1}{2}}(1).$$

and

$$\partial f = \cap_{\varepsilon > 0} (\partial f)^\varepsilon.$$

(EP) *Find an enlargement T' of T such that $0 \in T'(x)$ for all $x \in \operatorname{dom} T'$*

Suppose that T' is a solution of **(EP)**.

For $\epsilon \in [0,1]$, define $T_\epsilon : X \rightrightarrows X^*$ as

$$T_\epsilon(x) := \epsilon T'(x) + (1-\epsilon)T(x) \quad \text{for every} \quad x \in \operatorname{dom} T$$

and

$$T_\epsilon(x) := T'(x) \quad \text{for every} \quad x \in \operatorname{dom} T' \setminus \operatorname{dom} T.$$

Obviously $T(x) \subset T_\epsilon(x) \subset T'(x)$ for all $x \in X$ and all $0 \le \epsilon \le 1$.

Let $A \subset X$ be a given nonempty subset of X with $A \cap \operatorname{dom} T' \neq \emptyset$.

Since $0 \in T'(x)$ for all $x \in \operatorname{dom} T'$, thus there exists $0 \le \epsilon_0 \le 1$ such that

$$\inf\{\epsilon \in [0,1] : \ \exists x \in A \subset X : \ 0 \in T_\epsilon(x)\} = \epsilon_0.$$

If A satisfies a compactness property and T' satisfies some continuity properties, there exist a sequence $(x_n) \in A$ and a decreasing sequence $(\epsilon_n) \in]0,1]$ converging to ϵ_0 such that $0 \in T_{\epsilon_n}(x_n)$.

• If $\epsilon_0 = 0$ (and therefore $T_{\epsilon_0} = T$), then we can formulate the following Auxiliary Problem:

$$\boxed{\text{Find } x \in A \text{ such that } 0 \in T_{\epsilon_n}(x),} \tag{AP1}$$

which may be more tractable and somewhat easier to handle. In this case x_n is a solution of **(AP1)** and, under the compactness and continuity assumptions, the sequence (x_n) has a subsequence which converges to some solution $\bar{x} \in A$ of **(OP)**.

- If $\epsilon_0 > 0$, then **(OP)** fails to have a solution in A. In this case, instead of A, we can consider an increasing sequence of closed subsets $A_n \subset X$ $(A_1 \subset A_2 \subset \cdots)$ with $X = \bigcup A_n$ such that $A_1 \bigcap \operatorname{dom} T' \neq \emptyset$ (implying $A_n \bigcap \operatorname{dom} T' \neq \emptyset$ for all $n \in \mathbb{N}$).

Then $\varepsilon_n := \inf\{\epsilon \in [0,1] : \ \exists x \in A_n$ such that $T_\epsilon(x)\}$ is a decreasing sequence in $[0,1]$ (converging to zero), and we can formulate another Auxiliary Problem:

$$\boxed{\textit{Find } x_n \in \operatorname{dom} T_{\epsilon_n} \textit{ such that } 0 \in T_{\epsilon_n}(x_n).} \tag{AP2}$$

Notice that $T' := X^*$ is a solution of **(EP)** but it is not appropriate for our purposes. Indeed, by letting $T' := X^*$, we have $T_\epsilon(x) = X^*$ for all $0 < \epsilon \leq 1$ and $T_\epsilon(x) = T$ for $\epsilon = 0$. Such an enlargement is useless.

Finding an enlargement T' (as a solution of (EP)), plays a key role in this procedure.

The main scope of this presentation is to find some close solutions (enlargements) of problem **(EP)** when T is supposed to be the subdifferential operator $T = \partial f$.

In this case, finding a close solution of problem **(EP)** means to find an enlargement T' of ∂f such that T' satisfies the fundamental properties of ∂f such as

convexity, weak*-closedness, weak*-compactness and above all T' possesses the two calculus rules of sum and scalar multiplication.

On one hand we define two enlargements $\partial_{sym} f$ and $\partial_{sup} f$ that are any such close solutions of problem **(EP)** for which the corresponding set-valued mapping T_ϵ also satisfies the fundamental properties of ∂f.

On the other hand, ∂_{sym} and ∂_{sup} can be applied to optimization problems directly.

For instance, we will show that the two subdifferential equations $\partial_{sup} f(x) = \{0\}$ and $\partial_{sup}(-f)(x) = \{0\}$ can help detecting the possible optima of an arbitrary function f (f can be nonsmooth and nonconvex!).

Indeed, if $\partial_{sup} f(x) = \{0\}$ for some $x \in X$, then x is a candidate for the function f to attain its maximum, and if $\partial_{sup}(-f)(x) = \{0\}$ for some $\bar{x} \in X$, then $\bar{x}$ is a candidate for the function f to attain its minimum (Proposition ([?]), Examples 2.2 and 2.3).

Outline

The subset $\mathcal{E} \subset \mathbb{B}_{X^\star}$ is said to be norm-generating, if for any $y \in X$ there exists $e^\star \in \mathcal{E}$ such that $|\langle e^\star, y\rangle| = \|y\|$.

Includes the canonical basis of $\mathbb{R}^n$.

The collection of all weak* closed norm-generating subsets of $\mathbb{B}_{X^\star}$ is denoted by $\mathcal{F}$. Given $\mathcal{E} \in \mathcal{F}$, set

$$\partial^{\mathcal{E}}_{sup} f(\bar{x}) \quad := \quad \left\{ x^\star \in X^\star : \langle x^\star, y\rangle \le \sup_{u^\star \in \mathcal{E}} f(\bar{x} + \tau_{u^\star}(y)y) - f(\bar{x}) \;\; \forall y \in X \setminus \{0\} \right\},$$

where

$$\tau_{u^\star}(y) := \left| \left\langle u^\star, \frac{y}{\|y\|} \right\rangle \right|$$

for $u^\star \in X^\star$ and $y \in X \setminus \{0\}$.

The Sup*-subdifferential of f at $\bar{x}$, denoted by $\partial^\star_{sup} f(\bar{x})$, is defined as

$$\partial^\star_{sup} f(\bar{x}) := \bigcap_{\mathcal{E} \in \mathcal{F}} \partial^{\mathcal{E}}_{sup} f(\bar{x}).$$

The Sup-subdifferential of f at $\bar{x}$ is defined as

$$\partial_{sup} f(\bar{x}) := \left\{ x \in X^\star : \langle x, y\rangle \le \sup_{0 \le t \le 1} f(\bar{x} + ty) - f(\bar{x}) : \forall y \in X \right\}.$$

Example (1)

Let $X := \ell^p(\mathbb{N})$ *with* $p \geq 1$, $C := \{x \in \ell^p(\mathbb{N}) : |x_n| \leq 2^{\frac{-2n}{p}}, \quad n = 1, 2, \cdots\}$ *and* $\bar{x} \in C$,

Define $f : C \to \mathbb{R}$ *by setting*

$$f(x) := \sum -(2^{\frac{-2n}{p}} + x_n)^{\frac{1}{2}}.$$

$$0 \in \partial^\star_{sup} f(\bar{x}).$$

Each summand in the definition of f

- *is continuous and convex and its absolute value is bounded from above by* $2^{\frac{-n}{p}+\frac{1}{2}}$.
- *the series is uniformly convergent*
- *this shows that* f *is continuous and convex.*

Let (e_k) *denote the canonical basis of* $\ell^p(\mathbb{N})$ *and* $\mathcal{E} \in \mathcal{F}$.

- *We claim that* $\mathcal{E}$ *contains either* e_k *or* $-e_k$ *for all* $k \in \mathbb{N}$.

Proposition (☼)

Let $\bar{x} \in \operatorname{dom} f$. The following assertions hold true:

(i) $\partial^{\mathcal{E}}_{sup} f(\bar{x})$ *is convex and weak***-closed for all* $\mathcal{E} \in \mathcal{F}$. *As a consequence* $\partial^{\star}_{sup} f(\bar{x})$ *and* $\partial_{sup} f(\bar{x})$ *are convex and weak***-closed;*

(ii) $0 \in \partial_{sup} f(\bar{x})$. *If* $\bar{x}$ *maximizes* f, *then* $\partial_{sup} f(\bar{x}) = \{0\}$*;*

(iii) *If* f *is convex, then* $\partial f(\bar{x}) \subset \partial^{\mathcal{E}}_{sup} f(\bar{x}) \subset \partial_{sup} f(\bar{x})$ *for all* $\mathcal{E} \in \mathcal{F}$. *As a consequence* $\partial f(\bar{x}) \subset \partial^{\star}_{sup} f(\bar{x}) \subset \partial_{sup} f(\bar{x})$*;*

(iv) *If* f *is convex, then* $\bar{x}$ *minimizes* f *if and only if* $\partial f(\bar{x}) = \partial_{sup} f(\bar{x})$. *As a consequence* $\bar{x}$ *minimizes* f *if and only if* $\partial f(\bar{x}) = \partial^{\star}_{sup} f(\bar{x}) = \partial_{sup} f(\bar{x})$*;*

(v) *If* f *is convex and* $\partial_{sup} f(\bar{x})$ *is a singleton, then either* $\bar{x}$ *minimizes* f *or* $\partial f(\bar{x}) = \emptyset$*;*

(vi) *Suppose that the function* $y \mapsto f(\bar{x} + y)$ *is bounded on* $\mathbb{B}_X$. *Then* $\partial^{\mathcal{E}}_{sup} f(\bar{x})$ *is weak***-compact for all* $\mathcal{E} \in \mathcal{F}$. *As a consequence* $\partial_{sup} f(\bar{x})$ *and* $\partial^{\star}_{sup} f(\bar{x})$ *are weak***-compact;*

(vii) *Suppose that* X *is finite dimensional and* f *is continuous. Then* $\partial^{\mathcal{E}}_{sup} f(\bar{x})$ *is compact for all* $\mathcal{E} \in \mathcal{F}$. *As a consequence* $\partial_{sup} f(\bar{x})$ *and* $\partial^{\star}_{sup} f(\bar{x})$ *are compact.*

Corollary (☼)

If $\bar{x} \in \operatorname{dom} f$ *maximizes* f, *then* $\partial_{sup} f(\bar{x}) = \{0\}$, *and if* $\bar{x}$ *minimizes* f, *then* $\partial_{sup}(-f)(\bar{x}) = \{0\}$.

Proof.

Apply Proposition (☼)(ii). □

Example (2)

$$f(x) := \begin{cases} x^2 - 1 & |x| \geq 1 \\ 2 - 2x^2 & 0 \leq x < 1 \\ x + 1 & -1 < x < 0. \end{cases}$$

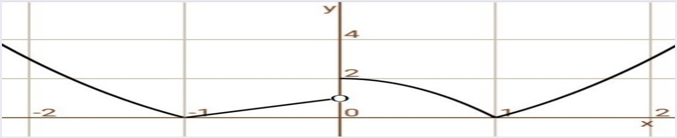

Figure: *The graph of f (Example 2)*

One has $\sup\limits_{0 \leq t \leq 1} f(ty) = 2$ *for all* $y \in]-1, 1[$*. Let* $a \in \partial_{sup} f(0)$*. Then we must have* $ay \leq \sup\limits_{0 \leq t \leq 1} f(ty) - f(0)$ *for all* $y \in]-1, 1[$ *which implies that* $a = 0$*. Hence,* $\partial_{sup} f(0) = \{0\}$*.*

One can also check that

- $\partial_{sup}(-f)(\pm 1) = \{0\}$,
- *while at all other points the sup-subdifferential of both* f *and* $-f$ *does not equal* $\{0\}$*.*

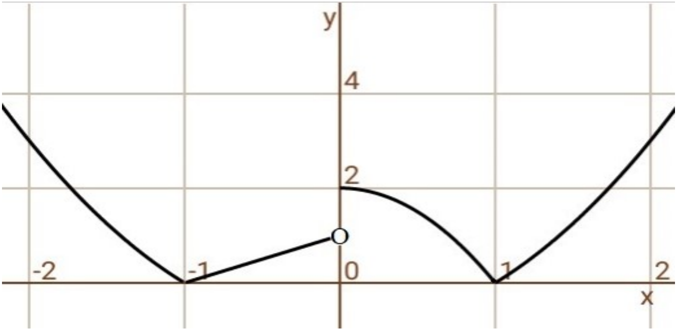

Figure: The graph of f (Example 2)

Hence, by Corollary (☁) ± 1 and 0 are the only candidates for the function f to attain its local minima and maxima, respectively. f is nonconvex and fails to be continuous at zero (although it is upper semi-continuous at 0) and $\partial f(0) = \emptyset$.

Example (3)

$$f(x) := \begin{cases} x^2 - 1 & |x| \geq 1 \\ 2x^2 - 2 & 0 < x < 1 \\ -x - 1 & -1 < x \leq 0. \end{cases}$$

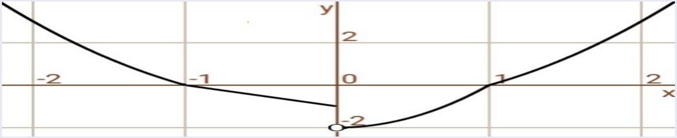

Figure: *The graph of f (Example 3)*

We have

$$\sup_{0 \leq t \leq 1} (-f)(ty) = \begin{cases} 2 & y \geq 0 \\ 1 & y < 0. \end{cases}$$

It follows that $\partial_{sup}(-f)(0) = \{0\}$, while at all other points the sup-subdifferential of $-f$ and f does not equal $\{0\}$. Hence, by Corollary (☀) 0 is the only candidate for the function f to attain its local minimum. But 0 fails to be a minimizer of f. One can check that $\partial f(0) = \emptyset$.

$$\partial_C f(\bar{x}) = \{x^\star : \langle x^\star, x\rangle \le f(\bar{x}+x) - f(\bar{x}) \quad \forall x \in C\} \qquad \partial_\emptyset f(\bar{x}) = X^\star$$

Theorem (▲)

Let f be upper semi-continuous and $\bar{x} \in \operatorname{dom} f$. Then for all $\mathcal{E} \in \mathcal{F}$

$$\partial^{\mathcal{E}}_{sup} f(\bar{x}) = \bigcap_{0<\lambda\le 1} \left\{\lambda \partial_{C^{\mathcal{E}}_\lambda} f(\bar{x})\right\} \bigcap \left(\tau_{\mathcal{E}}^{-1}(0)\right)^\circ$$

where $C^{\mathcal{E}}_\lambda := \lambda \tau_{\mathcal{E}}^{-1}(\lambda) \setminus \lambda \tau_{\mathcal{E}}^{-1}(0)$. Consequently,

$$\partial^\star_{sup} f(\bar{x}) = \bigcap_{\mathcal{E}\in\mathcal{F},\ \ 0<\lambda\le 1} \left\{\lambda \partial_{C^{\mathcal{E}}_\lambda} f(\bar{x})\right\} \bigcap \left(\tau_{\mathcal{E}}^{-1}(0)\right)^\circ,$$

and

$$\partial_{sup} f(\bar{x}) = \bigcap_{0<\lambda\le 1} \left\{\lambda \partial_{C_\lambda} f(\bar{x})\right\} \bigcap \left(\tau^{-1}(0)\right)^\circ$$

$$\tau_{\mathcal{E}}(y) \quad := \quad \min\left\{\tau_{x^\star}(y) : \ f(\bar{x} + \tau_{x^\star}(y)y) = \sup_{u^\star\in\mathcal{E}} f(\bar{x} + \tau_{u^\star}(y)y),\ x^\star \in \mathcal{E}\right\}$$

$$\tau(y) \quad := \quad \min\left\{\lambda \in [0,1] : \ f(\bar{x} + \lambda y) = \sup_{0\le t\le 1} f(\bar{x} + ty)\right\}.$$

$$C^{\mathcal{E}}_\lambda := \lambda \tau_{\mathcal{E}}^{-1}(\lambda) \setminus \lambda \tau_{\mathcal{E}}^{-1}(0)$$

Sup* and Sup-Subdifferentials of Upper Semi-Continuous Convex Functions

Set

$$\begin{aligned} L_f^{>}(\bar{x}) &:= \{y \in X : f(\bar{x}+y) > f(\bar{x})\};\\ L_f^{<}(\bar{x}) &:= \{y \in X : f(\bar{x}+y) < f(\bar{x})\};\\ L_f^{=}(\bar{x}) &:= \{y \in X : f(\bar{x}+y) = f(\bar{x})\};\\ L_f^{\leq}(\bar{x}) &:= \{y \in X : f(\bar{x}+y) \leq f(\bar{x})\}. \end{aligned}$$

The following proposition provides an explicit representation of the functions $\tau_{\mathcal{E}}$ and τ for an upper semi-continuous convex function.

Proposition ()

Let f be convex upper semi-continuous, $\mathcal{E} \in \mathcal{F}$ and $\bar{x} \in \operatorname{dom} f$. If $0 \in \mathcal{E}$, then

$$\tau_{\mathcal{E}}(y) = \begin{cases} 1 & \text{if } \; y \in L_f^{>}(\bar{x}) \\ 0 & \text{if } \; y \in L_f^{\leq}(\bar{x}) \end{cases}$$

for all $y \in X$. As a consequence

$$\tau(y) = \begin{cases} 1 & \text{if } \; y \in L_f^{>}(\bar{x}) \\ 0 & \text{if } \; y \in L_f^{\leq}(\bar{x}) \end{cases}$$

Corollary ($\bigtriangledown$)

Let f be convex upper semi-continuous, $\mathcal{E} \in \mathcal{F}$ and $\bar{x} \in \operatorname{dom} f$. If $0 \in \mathcal{E}$, then

$$\partial^{\mathcal{E}}_{sup} f(\bar{x}) = \partial_{L_f^{>}(\bar{x})} f(\bar{x}) \bigcap N_{L_f^{\leq}(\bar{x})}(0).$$

Therefore

$$\partial_{sup} f(\bar{x}) = \partial_{L_f^{>}(\bar{x})} f(\bar{x}) \bigcap N_{L_f^{\leq}(\bar{x})}(0).$$

As a consequence,

$$\bar{x} \text{ minimizes } \quad f \quad \Longleftrightarrow \quad \partial f(\bar{x}) = \partial_{L_f^{>}(\bar{x})} f(\bar{x}) \bigcap N_{L_f^{\leq}(\bar{x})}(0).$$

Corollary ($\triangle$)

Let f be convex upper semi-continuous and $\bar{x} \in \operatorname{dom} f$. Let $X := \ell^p(\mathbb{N})$ with $p \geq 1$. Then $\partial^{\star}_{sup} f(\bar{x}) = \partial_{sup} f(\bar{x})$. Consequently,

$$\bar{x} \text{ minimizes } \quad f \quad \Longleftrightarrow \quad \partial^{\star}_{sup} f(\bar{x}) = \partial f(\bar{x}).$$

We conclude this section adding only a few remarks about the sup-subdifferential. The subdifferential $\partial_{sup} f$ can be connected with a notion of directional derivative. Indeed, if f is convex, then the function

$$h \mapsto \frac{\sup_{0 \leq t \leq h} f(\bar{x}+ty) - f(\bar{x})}{h},$$

is nondecreasing, and the function

$$y \mapsto \lim_{h \downarrow 0} \frac{\sup_{0 \leq t \leq h} f(\bar{x}+ty) - f(\bar{x})}{h},$$

is positively homogeneous (note that the limit exists in $\mathbb{R} \cup \{\pm\infty\}$). Let us denote this limit by $f'_{sup}(\bar{x}; y)$. It follows that

$$\partial_{sup} f(\bar{x}) = \{x^\star \in X^\star : \ \langle x^\star, y \rangle \leq f'_{sup}(\bar{x}; y) : \ \forall y \in X\}.$$

It is worth mentioning that the subdifferential $\partial_{sup} f(\cdot)$ could be defined in any topological vector space.

Outline

Let X be a linear topological space and $\bar{x} \in \operatorname{dom} f$. The symmetric subdifferential of f at $\bar{x}$, denoted by $\partial_{sym} f(\bar{x})$, is defined as

$$\partial_{sym} f(\bar{x}) := \left\{ x^\star \in X^\star : \quad \langle x^\star, d \rangle \leq f'_{sym}(\bar{x}; d), \quad \forall d \in X \right\}$$

where

$$f'_{sym}(\bar{x}; d) := \lim_{t \downarrow 0} \frac{\max\{f(\bar{x} + td), f(\bar{x} - td)\} - f(\bar{x})}{t}$$

(if the limit exists in $\mathbb{R} \cup \{\pm\infty\}$). $f'_{sym}(\bar{x}; d)$ is referred to as the symmetric directional derivative of f at $\bar{x}$ in direction d. If f is convex, then $f'_{sym}(\bar{x}; d)$ exists in $\mathbb{R} \cup \{\pm\infty\}$ and is finite if $\bar{x} \in \operatorname{int} \operatorname{dom} f$. Indeed,

$$f'_{sym}(\bar{x}; d) = \max\left\{ f'(\bar{x}; d), f'(\bar{x}; -d) \right\}, \tag{1}$$

where $f'(\bar{x}; d)$ denotes the conventional directional derivative of f at $\bar{x}$ in direction $d \in X$. Note that, if the limit

$$\lim_{t \to 0} \frac{f(\bar{x} + ty) - f(\bar{x})}{t},$$

exists, then

$$f'_{sym}(\bar{x}; d) = |f'(\bar{x}; d)|. \tag{2}$$

Proposition (🐟)

Suppose that f is convex and continuous at $\bar{x}$. Then

$$f'_{sym}(\bar{x}; d) = \max\left\{\langle x^\star, d\rangle : \quad x^\star \in \partial f(\bar{x}) \cup \{-\partial f(\bar{x})\}\right\}.$$

Proposition (⌂)

If f is convex and for some $\bar{x} \in X$ one has $\partial_{sym} f(\bar{x}) = \partial f(\bar{x}) \neq \emptyset$, then $\bar{x}$ minimizes f.

The following theorem asserts that under a mild assumption, the symmetric subdifferential $\partial_{sym} f(\cdot)$ is nonempty.

Theorem (☺)

Let f be convex and $\bar{x} \in \operatorname{dom} f$. If there exists a direction $\bar{d} \in X$ such that

$$0 < \max\left\{f'(\bar{x}; \bar{d}), f'(\bar{x}; -\bar{d})\right\} < +\infty,$$

then $\partial_{sym} f(\bar{x})$ contains a nonzero element.

Example (4)

Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots) \in C$ *and* $\bar{d} = e_k$*, the kth basis vector for* $X := \ell^2(\mathbb{N})$ *for some* $k \in \mathbb{N}$*. One can easily check that*

$$f'(\bar{x};\bar{d}) = -\frac{1}{2}(\bar{x}_k + 2^{-k})^{-\frac{1}{2}}, \quad f'(\bar{x};-\bar{d}) = \frac{1}{2}(\bar{x}_k + 2^{-k})^{-\frac{1}{2}}.$$

Hence, $\partial_{sym} f(\bar{x})$ *contains a nonzero element.*

Example (5)

$$f(x) := \begin{cases} x & x > 0 \\ 1 - x & x \le 0. \end{cases}$$

f is not continuous at zero (although it is upper semi-continuous at 0) and fails to be convex. One can easily check that

$$\partial f(0) = \emptyset \quad \text{and} \quad \partial_{sym} f(0) = [-1, 1].$$

Indeed,

$$\max\left\{f'(0;d), f'(0;-d)\right\} = |d|.$$

Lemma (◑)

Let f be a convex function, continuous at $\bar{x}$. Assume that the function $y \mapsto f(\bar{x}+y)$ is bounded on $\mathbb{B}_X$. Then

$$\partial_{sym} f(\bar{x}) = cl^{w^\star} \operatorname{co}\left(\partial f(\bar{x}) \cup \{-\partial f(\bar{x})\}\right)$$

If X is Asplund (in the sense that, every convex continuous function on X is generically Fréchet differentiable), then

$$\partial_{sym} f(\bar{x}) = \operatorname{cl} \operatorname{co}\left(\partial f(\bar{x}) \cup \{-\partial f(\bar{x})\}\right)$$

Theorem (✠)

Let $A : X \to Y$ be a bounded linear mapping between Banach spaces. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions such that f and $g \circ A$ are continuous at $\bar{x}$. Suppose that $0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f)$. Assume further that the functions $x \mapsto f(\bar{x} + x)$ and $y \mapsto g(A\bar{x} + y)$ are bounded on $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively. Then

(i)

$$\partial_{sym}(f + g \circ A)(\bar{x}) \subset \partial_{sym} f(\bar{x}) + A^\star \partial_{sym} g(A\bar{x}).$$

Furthermore, if for any direction $d \in X$,

$$f'(\bar{x}; d) \geq f'(\bar{x}; -d) \implies g'(A\bar{x}; Ad) \geq g'(A\bar{x}; -Ad),$$

then

$$\partial_{sym}(f + g \circ A)(\bar{x}) = \partial_{sym} f(\bar{x}) + A^\star \partial_{sym} g(A\bar{x}).$$

(ii)

$$\partial_{sup}(f + g \circ A)(\bar{x}) \subset \partial_{sup} f(\bar{x}) + A^\star \partial_{sup} g(A\bar{x}).$$

Furthermore, if for any direction $d \in X$ we have

Theorem (◐)

Let $A : X \to Y$ be a bounded linear mapping between Banach spaces. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions such that f and $g \circ A$ are c ontinuous at $\bar{x}$. Suppose that $0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f)$. Assume further that the functions $x \mapsto f(\bar{x} + x)$ and $y \mapsto g(A\bar{x} + y)$ are bounded on $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively. Then

(i)

$$\partial_{sym}(f + g \circ A)(\bar{x}) \subset \partial_{sym} f(\bar{x}) + A^\star \partial_{sym} g(A\bar{x}).$$

Furthermore, if for any direction $d \in X$,

$$f'(\bar{x}; d) \geq f'(\bar{x}; -d) \implies g'(A\bar{x}; Ad) \geq g'(A\bar{x}; -Ad),$$

then

$$\partial_{sym}(f + g \circ A)(\bar{x}) = \partial_{sym} f(\bar{x}) + A^\star \partial_{sym} g(A\bar{x}).$$

(ii)

$$\partial_{sup}(f + g \circ A)(\bar{x}) \subset \partial_{sup} f(\bar{x}) + A^\star \partial_{sup} g(A\bar{x}).$$

Furthermore, if for any direction $d \in X$ we have

$$f'(\bar{x}; d) \geq 0 \iff g'(A\bar{x}; Ad) \geq 0,$$

then

$$\partial_{sup}(f + g \circ A)(\bar{x}) = \partial_{sup} f(\bar{x}) + A^\star \partial_{sup} g(A\bar{x}).$$

Theorem

Let $X := \ell^p(\mathbb{N})$, $p \geq 1$ *and* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be proper convex functions, continuous at* $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$. *Suppose that*

$$0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g).$$

Assume further that the functions $y \mapsto f(\bar{x} + y)$ *and* $y \mapsto g(\bar{x} + y)$ *are bounded on* $\mathbb{B}_X$. *Then*

$$\partial^\star_{sup}(f+g)(\bar{x}) \subset \partial^\star_{sup} f(\bar{x}) + \partial^\star_{sup} g(\bar{x}).$$

If for any direction $d \in X$ *we have*

$$f'(\bar{x}; d) \geq 0 \iff g'(\bar{x}; d) \geq 0,$$

then

$$\partial^\star_{sup}(f+g)(\bar{x}) = \partial^\star_{sup} f(\bar{x}) + \partial^\star_{sup} g(\bar{x}).$$

The next statement is a straightforward consequence of Theorem (✢) and Corollary (△).

Theorem (❀)

Let $f, g : \ell^p(\mathbb{N}) \to \mathbb{R}_\infty \quad (p \geq 1)$ *be proper convex functions, continuous at* $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$*. Suppose that* $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$*, and the functions* $x \mapsto f(\bar{x}+x)$ *and* $x \mapsto g(\bar{x}+x)$ *are bounded on* $\mathbb{B}_X$*. Then*

$$\partial^\star_{sup}(f+g)(\bar{x}) \subset \partial^\star_{sup} f(\bar{x}) + \partial^\star_{sup} g(\bar{x}).$$

If for any $d \in X$ *we have*

$$f'(\bar{x}; d) \geq 0 \Longleftrightarrow g'(\bar{x}; d) \geq 0,$$

then

$$\partial^\star_{sup}(f+g)(\bar{x}) = \partial^\star_{sup} f(\bar{x}) + \partial^\star_{sup} g(\bar{x}).$$

 

M. Abassi, A. K. Kruger, M.T, ***Enlargements of the Moreau-Rockafellar Subdifferential,*** *https://hal.archives-ouvertes.fr/hal-02484321*

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