A General Asymptotic Function with Applications in Nonconvex Optimization

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Optimization Seminars

The *lower effective domain* and the *upper effective domain* of $h : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$\operatorname{dom}_L h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$$
$$\operatorname{dom}_U h := \{x \in \mathbb{R}^n : h(x) > -\infty\}.$$

We say that h is *lower proper* if $\operatorname{dom}_L h \neq \emptyset$ and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$. Similarly, h is *upper proper* if $\operatorname{dom}_U h \neq \emptyset$ and $h(x) < +\infty$ for all $x \in \mathbb{R}^n$. A function is said to be *proper*, if it is either lower proper or upper proper.

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The *effective domain* of any function h is, by definition,
$\operatorname{dom} h := \operatorname{dom}_L h \cap \operatorname{dom}_U h = h^{-1}(\mathbb{R})$. It is clear that if h is lower proper (respectively upper proper), then $\operatorname{dom} h = \operatorname{dom}_L h$ ($\operatorname{dom} h = \operatorname{dom}_U h$).
Note: A proper convex function is lower proper; a proper concave function is upper proper.

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The *hypograph* of h is the set $\operatorname{hyp} h = \left\{(x,t) \in \mathbb{R}^{n+1} : h(x) \geq t\right\}$.

The asymptotic cone of a set $K \subseteq \mathbb{R}^n$ is defined by

$$K^\infty := \left\{ u \in \mathbb{R}^n : \ \exists\, t_k \to +\infty, \ \exists\, x_k \in K, \ \frac{x_k}{t_k} \to u \right\}.$$

It is known that if K is convex and closed, then for every $x_0 \in K$,

$$K^\infty = \{ u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall\, \lambda \geq 0 \}.$$

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Given a lower proper function h, its asymptotic function $h^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$\operatorname{epi} h^\infty := (\operatorname{epi} h)^\infty.$$

Similarly, for an upper proper function h, the upper asymptotic function $h_U^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$\operatorname{hyp} h_U^\infty := (\operatorname{hyp} h)^\infty.$$

It follows that

$$h^{\infty}(u) = \inf \left\{ \liminf_{k \to +\infty} \frac{h(x_k)}{t_k} \,:\, t_k \to +\infty,\ \frac{x_k}{t_k} \to u \right\},$$

$$h_U^{\infty}(u) = \sup \left\{ \limsup_{k \to +\infty} \frac{h(x_k)}{t_k} \,:\, t_k \to +\infty,\ \frac{x_k}{t_k} \to u \right\}.$$

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Moreover, when h is (lower) proper, lower semicontinuous and convex, one has

$$h^{\infty}(u) = \sup_{t>0} \frac{h(x_0+tu)-h(x_0)}{t} = \lim_{t\to+\infty} \frac{h(x_0+tu)-h(x_0)}{t},$$

and when h is (upper) proper, upper semicontinuous and concave, one has

$$h^{\infty}_U(u) = \inf_{t>0} \frac{h(x_0+tu)-h(x_0)}{t} = \lim_{t\to+\infty} \frac{h(x_0+tu)-h(x_0)}{t},$$

Lower (resp. upper) semicontinuity are not needed if $x_0 \in \operatorname{ri}\operatorname{dom} h$.

A general asymptotic function

Definition

Given a proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, its general asymptotic function $h^G : \mathbb{R}^n \to \overline{\mathbb{R}}$ at direction $u \in \mathbb{R}^n$ is defined by

$$h^G(u) := \sup_{x \in \operatorname{dom} h} \inf_{t>0} \frac{h(x+tu) - h(x)}{t}.$$

Note:

- $h^G(0) = 0$, so $\operatorname{dom} h \neq \emptyset$;
- h^G is positively homogenous, so $\operatorname{dom} h$ is a cone.

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Note: there is no symmetry between the results concerning max-min, convexity-concavity, quasiconvexity-quasiconcavity, boundedness from above-below etc.

Proposition

Let h be proper. The following assertions hold:

(i) *If h is continuous and positively homogeneous, then $h^G = h$.*

(ii) $(h+c)^G = h^G$ *for every* $c \in \mathbb{R}$ *and* $(\lambda h)^G = \lambda h^G$ *for every* $\lambda > 0$.

(iii) *If h is a quadratic function of the form $h(x) = \frac{1}{2}\langle Ax, x\rangle + \langle x, a\rangle + \alpha$, where A is a symmetric n-matrix, $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then for every $u \in \mathbb{R}^n$,*

$$h^G(u) = \begin{cases} -\infty, & \text{if } \langle Au, u\rangle < 0, \\ +\infty, & \text{if } \langle Au, u\rangle \geq 0 \text{ and } Au \neq 0, \\ \langle a, u\rangle, & \text{if } Au = 0. \end{cases}$$

Proposition

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper function.*

- (i) *If* h *is convex, then* h^G *is convex. If in addition,* h *is lower semicontinuous, then* $h^G = h^\infty$.
- (ii) *If* h *is concave, then* $h^G = h^\infty_U$. *In particular,* h^G *is concave.*

Lower semicontinuity cannot be dropped in (i).

Proposition

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(ii) *If h is concave, then $h^G = h^\infty_U$. In particular, h^G is concave.*

Lower semicontinuity cannot be dropped in (i).

A useful formula:

Proposition

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function which is either lower proper and lower semicontinuous, or upper proper and concave. Then

$$h^G(u) = \sup_{x \in \operatorname{dom} h} \liminf_{t \to +\infty} \frac{h(x+tu) - h(x)}{t}, \ \forall\, u \in \mathbb{R}^n.$$

Of course, $\liminf_{t\to+\infty} \frac{h(x+tu)-h(x)}{t} \neq \inf_{t>0} \frac{h(x+tu)-h(x)}{t}$ in general.

Relation to boundedness

Proposition

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Then the following assertions hold:

(i) *If h is bounded from above, then $h^G(u) \leq 0$ for all $u \in \mathbb{R}^n$.*

(ii) *If h is bounded from below and is either lower semicontinuous or quasiconcave, then $h^G(u) \geq 0$ for all $u \in \mathbb{R}^n$.*

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- The lower semicontinuity or the quasiconcavity assumption cannot be dropped in (ii).
- The converse statements do not hold.

A sufficient condition for $h^G \equiv 0$:

Proposition

Let h be lower proper and lower semicontinuous. Then

$$\lim_{\|x\| \to +\infty} \frac{h(x)}{\|x\|} = 0 \implies h^G(u) = 0, \ \forall \, u \in \mathbb{R}^n.$$

The converse holds for convex functions, but it does not hold in general, even if h is quasiconvex and continuous.

Relation to other asymptotic functions

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be lower proper. Its *q-asymptotic function* is defined by

$$h_q^\infty(u) := \sup_{x \in \operatorname{dom} h} \sup_{t>0} \frac{h(x+tu) - h(x)}{t}, \; u \in \mathbb{R}^n$$

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- $h^\infty(u) \leq h_q^\infty(u)$ for all $u \in \mathbb{R}^n$; equality holds when h is lower semicontinuous and convex.
- if h is lower semicontinuous and quasiconvex, then $\operatorname{argmin} h \neq \emptyset$ and bounded iff $h_q^\infty(u) > 0$ for all $u \neq 0$.

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- h_q^∞ is always convex (!)

Proposition

If $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a lower proper function, then $h^G(u) \le h_q^\infty(u)$ for all $u \in \mathbb{R}^n$. If in addition h is lower semicontinuous, then $h^\infty(u) \le h^G(u)$ for all $u \in \mathbb{R}^n$.

Elementary calculus rules

Let $h_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in I$. Denote $H := \sup_{i\in I} h_i$ and $h := \inf_{i \in I} h_i$.

Proposition

Assume that h_i, $i \in I$, are lower proper and lower semicontinuous.

(*i*) *If* $\operatorname{dom} H = \operatorname{dom} h_i$ *for all* $i \in I$, *then*

$$\left(\sup_{i\in I} h_i\right)^G \geq \sup_{i\in I} (h_i)^G.$$

(*ii*) *If* I *is finite and* $\operatorname{dom} h = \operatorname{dom} h_i$ *for all* $i \in I$, *then*

$$\left(\inf_{i\in I} h_i\right)^G \leq \inf_{i\in I} (h_i)^G.$$

Proposition

Let h_i, $i \in I$ be a collection of proper functions.

(i) *Assume that* $\sup_{i\in I} h_i$ *is proper. If all* h_i *are convex and lower semicontinuous, then*

$$\left(\sup_{i\in I} h_i\right)^G = \sup_{i\in I} (h_i)^G.$$

(ii) *Assume that* $\inf_{i\in I} h_i$ *is proper. If all* h_i *are concave and upper semicontinuous, then*

$$\left(\inf_{i\in I} h_i\right)^G = \inf_{i\in I} (h_i)^G.$$

Proposition

Let $h:\mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and* $A:\mathbb{R}^n \to \mathbb{R}^m$ *be linear. Define* $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ *by* $f = h \circ A$ *and assume that* $A(\mathbb{R}^n) \cap \operatorname{dom} h \neq \emptyset$. *Then*

$$f^G(u) \leq h^G(Au), \ \forall\, u \in \mathbb{R}^n. \tag{1}$$

If $\operatorname{dom} h \subseteq A(\mathbb{R}^n)$, *then* $f^G(u) = h^G(Au)$.

- Lower semicontinuity of h is not necessary.
- Equality also holds when h is lower semicontinuous and convex (or upper semicontinuous and concave), without the domain of h lying in the range of A.

Boundedness of $\operatorname{argmin} h$, $\operatorname{argmax} h$

In the lsc convex (usc concave) case we have $h^G = h^\infty$ ($h^G = h^U$), so

Proposition

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function.

(i) *If h is lower proper, lower semicontinuous and convex, then*

$$h^G(u) > 0,\ \forall\, u \neq 0 \iff \operatorname{argmin} h \neq \emptyset \text{ and compact.}$$

(ii) *If h is upper proper, upper semicontinuous and concave, then*

$$h^G(u) < 0,\ \forall\, u \neq 0 \iff \operatorname{argmax} h \neq \emptyset \text{ and compact.}$$

What happens beyond the classes of convex or concave functions?

Theorem

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be given. The following statements hold.*

(i) *If h is lower proper, lower semicontinuous and quasiconvex, and if*

$$h^G(u) > 0, \ \forall\, u \neq 0,$$

then h is lower coercive. As a consequence, $\operatorname{argmin} h$ *is nonempty and compact.*

(ii) *If h is upper proper, quasiconcave, and if*

$$h^G(u) < 0, \ \forall\, u \neq 0,$$

then h is upper coercive. As a consequence, if in addition, h is upper semicontinuous, then $\operatorname{argmax} h$ *is nonempty and compact.*

- The converse statements do not hold.
- The assumptions of lower semicontinuity and quasiconvexity cannot be dropped.
- For case (i) we know that $h^{\infty} \leq h^{G} \leq h_{q}^{\infty}$. We also know that $h_{q}^{\infty}(u) > 0$, $\forall u \neq 0$ is equivalent to $\operatorname{argmin} h$ being nonempty and compact. This partly proves (i). On the other hand, it is also known that $h^{\infty}(u) > 0, \forall u \neq 0$ implies that h is lower coercive. However, $h^{G}(u) > 0, \forall u \neq 0$ does not imply $h^{\infty}(u) > 0, \forall u \neq 0$.
- The theorem does not hold for the classical asymptotic functions.

Quadratic Optimization

We consider the following quadratic problem (QP),

$$\begin{aligned} &\min \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha \\ &\text{s.t. } x \in K, \end{aligned}$$

where A is a symmetric n-matrix, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and K is a closed convex set in $\mathbb{R}^n$. If A is positive definite, then the objective function of (QP) is strictly convex, so the set of optimal solutions is nonempty and bounded. What happens if A is not necessarily positive definite?

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We compute the G-asymptotic function of the function

$$h(x) := \begin{cases} \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha, & \text{if } x \in K, \\ +\infty, & \text{elsewhere,} \end{cases}$$

Lemma

The G-asymptotic function of h is given by

$$h^G(u) = \begin{cases} +\infty, & \text{if either } u \notin K^\infty \text{ or } u \in K^\infty, \langle Au, u\rangle > 0, \\ -\infty, & \text{if } u \in K^\infty \text{ and } \langle Au, u\rangle < 0, \\ \langle a, u\rangle + \sup_{x \in K}\langle Ax, u\rangle, & \text{if } u \in K^\infty \text{ and } \langle Au, u\rangle = 0. \end{cases}$$

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Based on this, one can show:

Proposition

The set of optimal solutions of (QP) is nonempty and compact provided that either $\langle Au, u\rangle > 0$ *for all* $u \in K^\infty \backslash \{0\}$, *or h is quasiconvex,* $\langle Au, u\rangle \geq 0$ *for all* $u \in K^\infty$ *and whenever* $\langle Au_0, u_0\rangle = 0$ *for some* $u_0 \in K^\infty \backslash \{0\}$, *there exists* $x \in K$ *such that* $\langle Ax + a, u_0\rangle > 0$.

The quasiconvexity of h cannot be omitted in the second condition of Proposition.

A necessary and sufficient condition for h to be merely quasiconvex (that is, quasiconvex but not convex) on a set K with nonempty interior is the following: A has exactly one negative eigenvalue λ_1; there exists $s \in \mathbb{R}^n$ such that $As + a = 0$; and $K \subseteq s \pm T$. Here, T is the cone

$$T := \{x : \langle Ax, x\rangle \leq 0, \ \langle x, v\rangle \geq 0\}.$$

where v is an eigenvector of A corresponding to λ_1 whose first component is nonnegative.

Quadratic Fractional Programming

We consider the following quadratic fractional optimization problem (QFP):

$$\min \frac{\frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha}{\langle b, x\rangle + \beta}$$
$$\text{s.t. } x \in K$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $a, b \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, and $K := \{x \in \mathbb{R}^n : \ g(x) \geq \gamma\}$ ($\gamma > 0$ fixed). Define the function h by

$$h(x) = \begin{cases} \frac{\frac{1}{2}\langle Ax,x\rangle + \langle a,x\rangle + \alpha}{\langle b,x\rangle + \beta}, & x \in K \\ +\infty, & x \notin K \end{cases}$$

Computation of h^G

Note that $K^\infty = \{u \in \mathbb{R}^n : \langle b, u \rangle \geq 0\}$.

Lemma

Let $u \in K^\infty$. Then the G-asymptotic function of the quadratic fractional function h is given by:

$$h^G(u) = \begin{cases} \dfrac{\langle Au, u \rangle}{\langle b, u \rangle}, & \text{if } \langle b, u \rangle > 0, \\ \sup\limits_{x \in K} \dfrac{\langle Ax + a, u \rangle}{\langle b, x \rangle + \beta}, & \text{if } \langle b, u \rangle = 0 \text{ and } \langle Au, u \rangle = 0, \\ +\infty, & \text{if } \langle b, u \rangle = 0 \text{ and } \langle Au, u \rangle > 0, \\ -\infty, & \text{if } \langle b, u \rangle = 0 \text{ and } \langle Au, u \rangle < 0. \end{cases}$$

One can now show:

Proposition

Suppose that h is a pseudoconvex quadratic fractional function over K. Then the following conditions are necessary and sufficient for h to be lower coercive:

(i) $\langle Au, u\rangle \geq 0$ *for all* $u \in K^\infty$*;*

(ii) $\{u \in K^\infty : \langle Au, u\rangle = 0,\ \langle b, u\rangle > 0\} = \emptyset$.

(iii) *For each* $u \in K^\infty \setminus \{0\}$ *with* $\langle Au, u\rangle = 0$ *and* $\langle b, u\rangle = 0$*, there is some* $x_0 \in K$ *such that* $\langle Ax_0 + a, u\rangle > 0$.

In particular, if A is semidefinite positive and conditions (ii) *and* (iii) *hold, then the set of optimal solutions of (QFP) is nonempty and compact.*

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